

ON THE DIRECTIONAL SYMMETRY OF THE IMPEDANCE*

S. A. HEIFETS†

*Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94309*

The independence of the impedance on the beam direction is an important feature of an accelerator structure, in particular, for the electron-positron storage rings where bunches of opposite charges travel through the same vacuum chamber in opposite directions. Recently Gluckstern and Zotter¹ considered a cylindrically symmetric but longitudinally asymmetric cavity with side pipes of equal radii. They were able to prove that for a relativistic particle the longitudinal impedance of the cavity with an arbitrary shape is independent of the direction in which the beam travels through it. Their result corroborates numerical observations of the independence of the wakefield obtained with the code TBCI. Bisognano² gave an elegant proof of the same statement. His approach is based on a reciprocity relation applied to the tensor Green's function.

I follow here his idea in a somewhat simpler way to obtain more general and physically transparent proof of this property for both *longitudinal* and *transverse* impedances. The result is valid for a cavity with no azimuthal symmetry and for arbitrary particle velocity, as soon as it may be considered constant. At the same time the limits of its validity (the side beam pipes must have the same cross sections) are shown.

Throughout this note it is assumed that the particle energy is constant and does not change as the result of radiation in the structure. Further, small oscillations that a particle performs while moving in an accelerator are neglected. In other words, it is assumed that the particle velocity $\mathbf{v}$ is constant (at least while traveling through a structure under consideration) and has only one component v_z along the axis.

Consider a cavity of an arbitrary configuration and let a bunch of a charge travel through it along the axis z . We attach a subscript 1(2) to all the quantities describing the case q when the bunch travels in a positive (negative) direction parallel to the axis z . To prove the theorem for both the longitudinal and transverse impedances we assume that the bunch trajectory is offset from the axis by a distance $|\mathbf{R}_\perp|$.

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†Visitor from CEBAF.

There is only z -component of the current density. The Fourier harmonics for the two cases are:

$$j_{1\omega z} = q\delta(\mathbf{r}_\perp - \mathbf{R}_\perp) \exp\left(\frac{i\omega z}{v}\right) , \quad \rho_{1\omega} = \frac{j_{1\omega z}}{v} , \quad (1)$$

$$j_{2\omega z} = -q\delta(\mathbf{r}_\perp - \mathbf{R}_\perp) \exp\left(-\frac{i\omega z}{v}\right) , \quad \rho_{2\omega} = -\frac{j_{2\omega z}}{v} . \quad (2)$$

Note that:

$$\mathbf{j}_{2\omega} = -\mathbf{j}_{1\omega}^* , \quad \rho_{2\omega} = \rho_{1\omega}^* . \quad (3)$$

The fields $\mathbf{E}_{1\omega}$ and $\mathbf{E}_{2\omega}$ excited by such sources satisfy the following wave equations:

$$\left(\Delta + \frac{\omega^2}{c^2}\right) \mathbf{E}_{1\omega} = 4\pi\nabla\rho_{1\omega} - \frac{4\pi i\omega}{c^2} \mathbf{j}_{1\omega} , \quad (4)$$

$$\left(\Delta + \frac{\omega^2}{c^2}\right) \mathbf{E}_{2\omega} = 4\pi\nabla\rho_{2\omega} - \frac{4\pi i\omega}{c^2} \mathbf{j}_{2\omega} , \quad (5)$$

as well as the boundary conditions for their tangential components:

$$\mathbf{E}_{1\omega}|_{tan} = 0 , \quad \mathbf{E}_{2\omega}|_{tan} = 0 , \quad (6)$$

and the radiation conditions for the radiated part of the EM field:

$$\begin{aligned} \lim_{z \rightarrow \pm\infty} \mathbf{E}_1^{rad}(z, \mathbf{R}_\perp, t) &= 0 , \\ \lim_{z \rightarrow \pm\infty} \mathbf{E}_2^{rad}(z, \mathbf{R}_\perp, t) &= 0 , \end{aligned} \quad (7)$$

for $z \rightarrow \pm\infty$. The longitudinal impedances Z_1 and Z_2 are^{3,4}:

$$Z_1(\omega, \mathbf{R}_\perp) = -\frac{1}{q} \int_{-\infty}^{\infty} dz E_{1z\omega}(\mathbf{R}_\perp, z) \exp\left(-\frac{i\omega z}{v}\right) , \quad (8)$$

$$Z_2(\omega, \mathbf{R}_\perp) = +\frac{1}{q} \int_{-\infty}^{\infty} dz E_{2z\omega}(\mathbf{R}_\perp, z) \exp\left(\frac{i\omega z}{v}\right) . \quad (9)$$

Let us show that the fields $\mathbf{E}_{1\omega}$ and $\mathbf{E}_{2\omega}$ are complex conjugate. Indeed, by substituting Eq. (3) into Eq. (4) and taking its complex conjugate, one obtains:

$$\left(\Delta + \frac{\omega^2}{c^2}\right) \mathbf{E}_{2\omega}^* = 4\pi\nabla\rho_{1\omega} - \frac{4\pi i\omega}{c^2} \mathbf{j}_{1\omega} \quad . \quad (10)$$

The boundary conditions shown in Eq. (6) are also valid for $\mathbf{E}_{2\omega}^*$. We now need only one additional assumption: that the synchronous parts of both fields $\mathbf{E}_{1\omega}$ and $\mathbf{E}_{2\omega}$ are the same at infinity. This is the case when the side pipes have the same cross sections (at least at infinity). If this assumption is true, then the equations and all the boundary conditions for $\mathbf{E}_{2\omega}^*$ and $\mathbf{E}_{1\omega}$ are the same, and we may conclude that:

$$\mathbf{E}_{2\omega}^* = \mathbf{E}_{1\omega} \quad . \quad (11)$$

From the Maxwell equation $i(\omega/c)\mathbf{B}_{1,2} = \nabla \times \mathbf{E}_{1,2}$ it follows that:

$$\mathbf{B}_{1\omega}^* = -\mathbf{B}_{2\omega} \quad . \quad (12)$$

Now multiply Eq. (4) by $\mathbf{E}_{2\omega}$, and Eq. (5) by $\mathbf{E}_{1\omega}$; subtract the results and integrate over the volume of the cavity and the side pipes bounded by imaginary cross sections at $z = \pm\xi, \xi \rightarrow \infty$. One then obtains the *Lorentz reciprocity theorem*⁵:

$$\frac{4\pi}{c} \int dV (\mathbf{E}_{2\omega} \cdot \mathbf{j}_{1\omega} - \mathbf{E}_{1\omega} \cdot \mathbf{j}_{2\omega}) = \int d\mathbf{S} \cdot (\mathbf{E}_{1\omega} \times \mathbf{B}_{2\omega} - \mathbf{E}_{2\omega} \times \mathbf{B}_{1\omega}) \quad . \quad (13)$$

The integration on the right-hand side is performed over the surface enclosing the volume over which the integration on the left-hand side is performed; i.e., over the walls of the cavity, the walls of the side pipes, and the bounding cross sections. Since the tangential electric field on the wall is zero, it is sufficient to perform the integration over these cross sections only. The integration over the transverse coordinates in the left side of Eq. (13) is performed easily. The remaining integration over z gives the longitudinal impedance; cf., Eqs. (8) and (9). We obtain the following expression for the difference of the impedances for two directions of the bunch travel:

$$\begin{aligned} \frac{4\pi q^2}{c} (Z_2(\omega, \mathbf{R}_\perp) - Z_1(\omega, \mathbf{R}_\perp)) &= \int d\mathbf{S} \cdot [(\mathbf{E}_{1\omega} \times \mathbf{B}_{2\omega} - \mathbf{E}_{2\omega} \times \mathbf{B}_{1\omega})_L \\ &- (\mathbf{E}_{1\omega} \times \mathbf{B}_{2\omega} - \mathbf{E}_{2\omega} \times \mathbf{B}_{1\omega})_R] \quad , \end{aligned} \quad (14)$$

where the subscripts R and L refer to the beampipe cross section at $z = \pm\xi$, respectively. Using Eqs. (11) and (12), this equation can be rewritten in the form:

$$\frac{4\pi q^2}{c} (Z_2(\omega, \mathbf{R}_\perp) - Z_1(\omega, \mathbf{R}_\perp)) = \int d\mathbf{S} \cdot [(\mathbf{E}_{1\omega} \times \mathbf{B}_{1\omega}^* + \mathbf{E}_{1\omega}^* \times \mathbf{B}_{1\omega})_R - (\mathbf{E}_{1\omega} \times \mathbf{B}_{1\omega}^* + \mathbf{E}_{1\omega}^* \times \mathbf{B}_{1\omega})_L] \quad (15)$$

The right-hand side of this equation is real. Hence, the imaginary parts of the impedances are equal:

$$\text{Im } Z_1(\omega, \mathbf{R}_\perp) = \text{Im } Z_2(\omega, \mathbf{R}_\perp) \quad (16)$$

The integrals in the right-hand side of Eq. (15) have the simple physical meaning of the electromagnetic (EM) field energy flowing through the cross sections of the side pipes. If these cross sections are far enough from the cavity, then the only part of the EM field present on them is the synchronous component accompanying the bunch. This is a direct consequence of the radiation condition [Eq. (7)] which is assumed to be fulfilled here. For the case when both side pipes have similar and equal cross sections, the synchronous components of the field at $z = \pm\infty$ are the same. It follows then from Eq. (15) that both longitudinal impedances are equal. Applying now the Panofsky-Wenzel theorem, we see that the same is also true for the transverse impedances.

However, for unequal or nonsimilar pipe cross sections, the synchronous components of the two fields are different, even at $z = \pm\infty$. We cannot say that Eqs. (11) and (12) are necessarily true, and the real parts of the impedances for two directions differ by a constant.

In the ultrarelativistic case $\gamma \rightarrow \infty$ for the side pipes with round cross sections, the difference of the energy of the synchronous components in the pipes with radii a and b is proportional to $\ln(b/a)$. Hence, the difference of the real parts of the impedances are proportional to the same constant. An example of such a case is given in a paper⁶ where an abrupt change in the cross section ("step") in a cylindrically symmetric pipe is considered. The impedances "in" and "out" are derived from the solution of a truncated system of the exact equations for the EM fields.

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