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On the Coupling Impedance of a Hole or Slot

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Abstract

Both the longitudinal and transverse coupling impedance produced by a small hole in the chamber walls are analytically evaluated at frequencies below cut-off. The method developed is based on the Bethe theory of diffraction by small holes. The estimates of the contribution from such elements to the coupling impedance of the UNK and LHC vacuum chamber are obtained.

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1 Introduction

The problem of the beam interaction with the vacuum chamber is of great importance in modern high-intensity accelerators and storage rings. This interaction is usually studied in terms of the beam-chamber coupling impedances or of wake potentials, see, for example, [1].

There is a general tendency to minimize the coupling impedances to avoid beam instabilities and reduce heating. In doing so one tends to shield, with respect to electromagnetic fields, all enlargements of the vacuum chamber, i.e. vacuum boxes, bellows, etc. On the other hand, the requirement of the beam life-time being long enough puts a constraint on the allowed value of vacuum inside the beam pipe. Achieving this high vacuum requires the presence of vacuum pumping holes or slots in the shields. The number of such elements can be very large in big machines. For example, so called liners in the UNK [2] have 26 slots per liner with the whole number of these liners in the ring being 3260. The LHC design [3] includes a thermal screen with nearly 10^7 small holes for pumping. So, the evaluation of the coupling impedances for these chamber elements is very important. Common wisdom tells us that these small discontinuities contribute very little to the longitudinal impedance since they do not interrupt the lines of beam-induced currents in the chamber walls. Nevertheless, since the number of such discontinuities can be very large, quantitative results should be obtained.

This paper presents the analytical calculation of both the longitudinal and transverse coupling impedance of a small hole or narrow slot. It is obvious that due to the absence of axial symmetry a numerical solution to the problem has to be essentially three-dimensional. This implies very time-consuming computations even in the case of a simplified model. On the other hand there is a small parameter in the problem, namely, the ratio of a typical hole size to a characteristic size of the chamber cross section. This circumstance allows us to apply the Bethe theory of diffraction by small holes [4] to the problem and obtain reasonable analytical estimates.

The paper is arranged as follows. In Section 2 we give the idea of the approach and evaluation of the longitudinal impedance. Section 3 deals with the transverse impedance. In Section 4 we discuss in short the problem of N holes and give some estimates for the UNK and LHC.

2 The longitudinal impedance

To evaluate the coupling impedance we have to calculate the fields induced in the chamber by a given current perturbation. In the problem examined this task can be split into two parts. First, it is easy to evaluate the fields produced by a given current, say, by a relativistic point charge, in the chamber without hole. Then we can consider these fields as incident electromagnetic waves on the hole. It was shown by H.A. Bethe in 1944 [4] that, when a plane e.m. wave is incident on an infinite conducting plane with a hole, the diffracted fields can be obtained as fields radiated by effective surface "magnetic" currents or, in the case of a small hole, simply by effective electric and magnetic dipoles. So, when this approach is applicable we can make the second step, namely, replace the excited hole by effective dipoles and look for the fields radiated by them inside the chamber. After evaluating these fields we easily obtain the coupling impedance.

Two remarks are in order at this point. First, the Bethe approach was used by M. Sands to estimate the energy loss from small holes for PEP [5]. He calculated the total power radiated by the effective dipoles and found that the energy loss was very small. Second, the application of the Bethe theory is the usual approach in the waveguide theory for evaluating the coupling of two waveguides by a small aperture, e.g., [6].

So, we will proceed in this Section as follows:

1. Calculate the fields produced by a charge on the hole.
2. Replace the excited hole by the effective dipoles.
3. Evaluate the fields produced by these dipoles inside the beam pipe.
4. Obtain the expression for the longitudinal impedance.

2.1 Beam fields in chamber without hole

Let us consider an infinite pipe with a circular cross section of radius b and perfectly conducting walls. The cylindrical coordinate system (r, φ, z) has the z axis directed along the pipe axis. Let a round hole with radius h $(h \ll b)$ be located at the point $(r = b, \varphi = \varphi_0, z = 0)$. The point charge q moves with velocity $v = c$ along the chamber axis with a transverse offset $\vec{a} = (r = a, \varphi = 0)$. Then the e.m. fields $\vec{E}^{(0)}, \vec{H}^{(0)}$, which would be produced by this charge in the chamber without hole, can be expressed as a series over azimuthal harmonics $\sin n\varphi$ and $\cos n\varphi$ ($a \le r \le b$, $\gamma \to \infty$)

$$\begin{aligned}
E_r^{(0)}(\vec{r}, z-ct) &= \frac{q}{2\pi\varepsilon_0}\delta(z-ct)\cdot \\
&\left\{\frac{1}{r} - \frac{1}{r}\ln\frac{b}{r} + \sum_{n=1}^{\infty}\frac{a^n}{n}\sin n\varphi\left[\frac{n-1}{r^{n+1}} + \frac{(n+1)r^{n-1}}{b^{2n}}\right]\right\} ; \\
Z_0 H_\varphi^{(0)}(\vec{r}, z-ct) &= \frac{q}{2\pi\varepsilon_0}\delta(z-ct)\cdot \\
&\left\{\frac{1}{r} + \sum_{n=1}^{\infty} a^n \cos n\varphi\left[\frac{1}{r^{n+1}} + \frac{r^{n-1}}{b^{2n}}\right]\right\} . \qquad (1)
\end{aligned}$$

All other components vanish on the wall. In the frequency domain ($f(\omega) = \int dt \exp(i\omega t) f(t)$) the field harmonics on the hole, i.e. in the point $(b, \varphi_0, 0)$, are

$$\begin{aligned}
E_r^{(0)}(b, \varphi_0, 0; \omega) &= \frac{Z_0 q}{2\pi b}\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n \sin n\varphi_0\right\} ; \\
H_\varphi^{(0)}(b, \varphi_0, 0; \omega) &= \frac{q}{2\pi b}\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n \cos n\varphi_0\right\} . \qquad (2)
\end{aligned}$$

2.2 Hole excitation

Since $h \ll b$ is assumed, one can neglect the wall curvature and consider the hole excitation by the fields (2) in the spirit of the Bethe theory. To be able to apply it, we assume also that the wall thickness is much smaller than the typical hole size, and restrict our consideration to the case when incident wave lengths are much larger than this size. To satisfy the boundary conditions on the hole, the effective surface "magnetic" charge density ρ_{mag} and current $\vec{J}_{mag}$ have to be introduced. The current and density values are related to the incident fields, Eq.(2). To calculate the fields produced by this current at distances $|\vec{R}|$ from the hole, which are much greater than h, one can replace the excited hole by effective dipoles. The effective magnetic dipole moment is defined [6] as

$$\vec{M} \equiv \frac{1}{\mu_0}\int\int_{hole} dS \rho_{mag} \vec{r} ,$$

and the electric one as

$$\vec{P} \equiv \frac{1}{2} \int\int_{hole} dS \vec{J}_{mag} \times \vec{r} \, .$$

It is very important that these effective moments are simply expressed in terms of the incident fields. Let us consider an elliptic hole with semiaxes l_1 and l_2 ($l_1 \geq l_2$), and introduce the local hole coordinates (u, v) with u along the major axis of the ellipse. The effective moments can be written [6] as

$$\begin{aligned} \vec{M} &= \vec{a}_u \alpha_{m\|} H_{\tau u}^{(0)} + \vec{a}_v \alpha_{m\perp} H_{\tau v}^{(0)} \, ; \\ \vec{P} &= \varepsilon_0 \alpha_e \vec{n} E_n^{(0)} \, , \end{aligned} \tag{3}$$

where $\vec{a}_u$ and $\vec{a}_v$ are the local coordinate unit vectors, $\vec{n}$ is the normal to the hole plane (u, v), $E_n^{(0)}$ is the normal component of the incident electric field on the hole and $\vec{H}_\tau^{(0)}$ is the tangential component of the incident magnetic field. For an elliptic hole the magnetic and electric polarizabilities α_m and α_e are given by [6]

$$\begin{aligned} \alpha_{m\|} &= \frac{\pi l_1^3 \varepsilon^2}{3[K(\varepsilon) - E(\varepsilon)]} \, , \\ \alpha_{m\perp} &= \frac{\pi l_1^3 \varepsilon^2 (1 - \varepsilon^2)}{3[E(\varepsilon) - (1 - \varepsilon^2) K(\varepsilon)]} \, , \\ \alpha_e &= -\frac{\pi l_1^3 (1 - \varepsilon^2)}{3E(\varepsilon)} \, , \end{aligned} \tag{4}$$

where $\varepsilon = \sqrt{1 - l_2^2/l_1^2}$ is the eccentricity, and $K(\varepsilon)$ and $E(\varepsilon)$ are complete elliptic integrals of the first and second kind. Let the angle between the chamber axis and the major axis of the ellipse be α. Since in our case $E_n^{(0)} = E_r^{(0)}$ and $\vec{H}_\tau^{(0)} = (0, H_\varphi^{(0)}, 0)$, see Eq.(2), it follows from (3) that

$$\begin{aligned} \vec{M} &= [\vec{a}_\varphi (\alpha_{m\|} \sin^2 \alpha + \alpha_{m\perp} \cos^2 \alpha) + \\ &+ \vec{a}_z (\alpha_{m\perp} - \alpha_{m\|}) \sin \alpha \cos \alpha] \; H_\varphi^{(0)} \, ; \\ \vec{P} &= \varepsilon_0 \alpha_e \vec{a}_r E_r^{(0)} \, , \end{aligned} \tag{5}$$

where $\vec{a}_r$, $\vec{a}_\varphi$ and $\vec{a}_z$ are the chamber coordinate unit vectors.

For the particular case of a circular hole with radius h, $\varepsilon = 0$ and we obtain from Eqs.(4)

$$\begin{aligned} \alpha_{m\|} &= \alpha_{m\perp} = \frac{4}{3}h^3 , \\ \alpha_e &= -\frac{2}{3}h^3 . \end{aligned} \tag{6}$$

For another important limit, a narrow slot in the longitudinal direction ($\alpha = 0$), with width w and length l, $w \ll l$, we have $\varepsilon \to 1$ and from Eqs.(4) [1]

$$\begin{aligned} \alpha_{m\perp} &= \frac{\pi}{24}lw^2 , \\ \alpha_e &= -\frac{\pi}{24}lw^2 . \end{aligned} \tag{7}$$

It should be noted that the mentioned above condition for applying the Bethe theory, namely, $\omega h/c \ll 1$, is not very restrictive due to the small size of the hole compared to the bunch length. For the case of a slot these conditions are $\omega w/c \ll 1$ and $\omega l/c \ll 1$. Moreover, to justify the far-region approximation for a slot, we have to restrict ourselves to slot lengths l which are not greater than b.

2.3 Fields radiated by hole in chamber

Now we are in the position to evaluate the e.m. fields radiated by the dipoles (3) inside the pipe. Let us expand these fields in a series over the waveguide eigenmodes [6]

$$\begin{aligned} \vec{F} &= \vec{F}^{+}\theta(z) + \vec{F}^{-}\theta(-z) = \\ &= \sum_{nm}\left(a_{nm}\vec{F}^{+}_{nm}\theta(z) + b_{nm}\vec{F}^{-}_{nm}\theta(-z)\right) , \end{aligned} \tag{8}$$

where $\vec{F}$ means either $\vec{E}$ or $\vec{H}$ and superscripts '+' or '−' denote fields radiated, respectively, in the positive ($z > 0$) or negative ($z < 0$) direction. In Eq.(8) $\theta(z)$ is the Heaviside step function and the unknown coefficients a_{nm} and b_{nm} have to be found. The summation in Eq.(8) runs over the complete

[1] This value, $\pi lw^2/24$, for a narrow slot is cited also in [7], but paper [5] and book [8] give another value, $\pi lw^2/16$.

set of the eigenfunctions. The fields $\vec{E}^{\pm}_{nm}$ and $\vec{H}^{\pm}_{nm}$ satisfy the homogeneous Maxwell equations and can be written in the form

$$\begin{aligned}\vec{E}^{\pm}_{nm} &= \left(\vec{e}^{\pm}_{nm} + (\vec{e}^{\pm}_{z})_{nm}\right)\exp\left(\mp\Gamma_{nm} z\right) , \\ \vec{H}^{\pm}_{nm} &= \left(\vec{h}^{\pm}_{nm} + (\vec{h}^{\pm}_{z})_{nm}\right)\exp\left(\mp\Gamma_{nm} z\right) . \end{aligned} \tag{9}$$

In Eqs.(9) $(\vec{e}^{\pm}_{z})_{nm} = 0$ for H-modes and $(\vec{h}^{\pm}_{z})_{nm} = 0$ for E-modes. The transverse eigenfunctions $\vec{e}^{\pm}_{nm}$ and $\vec{h}^{\pm}_{nm}$ form a complete orthogonal set and for a waveguide with circular cross section are well known, e.g. [6]. For instance, for E-modes with '+' sign which correspond to the propagation factor $\exp(-\Gamma_{nm} z)$ they are

$$\begin{aligned}\vec{e}_{nm}(r,\varphi) &= \vec{a}_r \frac{\mu_{nm}}{b} J'_n\left(\frac{\mu_{nm} r}{b}\right)\begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix} \\ &+ \vec{a}_\varphi \frac{n}{r} J_n\left(\frac{\mu_{nm} r}{b}\right)\begin{Bmatrix}\cos n\varphi \\ -\sin n\varphi\end{Bmatrix} ; \\ (e_z)_{nm}(r,\varphi) &= -\frac{\mu^2_{nm}}{\Gamma_{nm} b^2} J_n\left(\frac{\mu_{nm} r}{b}\right)\begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix} ; \\ \vec{h}_{nm}(r,\varphi) &= \frac{i\omega\varepsilon_0}{\Gamma_{nm}} \vec{a}_r \frac{n}{r} J_n\left(\frac{\mu_{nm} r}{b}\right)\begin{Bmatrix}\cos n\varphi \\ -\sin n\varphi\end{Bmatrix} \\ &- \frac{i\omega\varepsilon_0}{\Gamma_{nm}} \vec{a}_\varphi \frac{\mu_{nm}}{b} J'_n\left(\frac{\mu_{nm} r}{b}\right)\begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix} ; \end{aligned} \tag{10}$$

where $\Gamma^2_{nm} = \mu^2_{nm}/b^2 - k^2$; $k = \omega/c$; J_n are the Bessel functions of the first kind; $J_n(\mu_{nm}) = 0$, $n = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$. For '−' modes Γ_{nm} is just replaced by $-\Gamma_{nm}$. The eigenmode orthogonality conditions are

$$\begin{aligned} \iint_{S_\perp} \vec{e}_{nm} \cdot \vec{e}_{n'm'} &\sim \delta_{nn'}\delta_{mm'} ; \\ \iint_{S_\perp} \vec{h}_{nm} \cdot \vec{h}_{n'm'} &\sim \delta_{nn'}\delta_{mm'} ; \\ \iint_{S_\perp} \vec{a}_z \cdot \vec{e}_{nm} \times \vec{h}_{n'm'} &\sim \delta_{nn'}\delta_{mm'} . \end{aligned} \tag{11}$$

The integration in Eqs.(11) goes over a chamber cross section.

The e.m. fields (8) satisfy the inhomogeneous Maxwell equations, with the current $\vec{J}_{mag}$ in the RHS. By means of the Lorentz reciprocity theorem and with account of Eqs.(11) one can express the unknown coefficients in the expansion (8) in terms of $\vec{J}_{mag}$, see e.g. [6]:

$$2\left\{\begin{matrix} a_{nm} \\ b_{nm} \end{matrix}\right\} \int\int_{S_\perp} (e_\varphi^+ h_r^- - e_r^+ h_\varphi^-)_{nm} = \int\int_{hole} dS \vec{J}_{mag} \vec{H}_{nm}^{\mp} \ . \tag{12}$$

For a small hole the integral in the RHS can be expanded over the effective multipoles

$$\int\int_{hole} dS \vec{J}_{mag} \vec{H}_{nm}^{\mp} = -i\omega(\mu_0 \vec{H}_{nm}^{\mp} \vec{M} - \vec{E}_{nm}^{\mp} \vec{P} + \text{quads} + \ldots) \ , \tag{13}$$

in which $\vec{M}$ and $\vec{P}$ are given by Eqs.(3).

Then, taking into account the explicit form of the eigenmodes, Eq.(10), we rewrite Eq.(12) for E-modes as

$$\begin{aligned} a_{nm} &= \frac{\Gamma_{nm}}{2\varepsilon_0 \alpha_{nm}^2} \left(\mu_0 (h_\varphi^-)_{nm} M_\varphi - (e_r^-)_{nm} P_r\right) \ ; \\ b_{nm} &= \frac{\Gamma_{nm}}{2\varepsilon_0 \alpha_{nm}^2} \left(\mu_0 (h_\varphi^+)_{nm} M_\varphi - (e_r^+)_{nm} P_r\right) \ , \end{aligned} \tag{14}$$

where $e_r^{\pm}$ and $h_\varphi^{\pm}$ are taken at the hole, i.e. in the point (b, φ_0). In Eqs.(14) P_r and M_φ are defined by Eqs.(5) and the normalization constants are

$$\begin{aligned} \alpha_{nm}^2 &= \left\{\begin{matrix} 1+\delta_{n0} \\ 1-\delta_{n0} \end{matrix}\right\} \pi n^2 \int_0^b \frac{dr}{r} J_n^2 \left(\frac{\mu_{nm}}{b} r\right) \\ &+ \left\{\begin{matrix} 1-\delta_{n0} \\ 1+\delta_{n0} \end{matrix}\right\} \pi \frac{\mu_{nm}^2}{b^2} \int_0^b r dr J_n'^2 \left(\frac{\mu_{nm}}{b} r\right) \ . \end{aligned} \tag{15}$$

2.4 Impedance evaluation

The longitudinal impedance can be defined as

$$Z(\omega) = -\frac{1}{q} \int_{-\infty}^{\infty} dz e^{-ikz} E_z(r=0, z; \omega) \ , \tag{16}$$

where $k = \omega/c$. Substituting the field longitudinal component E_z at $r = 0$ from Eq.(8) gives

$$\begin{aligned} Z(\omega) &= -\frac{1}{qb^2} \sum_{m=1}^{\infty} \frac{\mu_{0m}^2}{\Gamma_{0m}} \\ &\cdot \int_{-\infty}^{\infty} dz e^{-ikz} \left[b_{0m} e^{\Gamma_{0m} z} \theta(-z) - a_{0m} e^{-\Gamma_{0m} z} \theta(z)\right] \ . \end{aligned} \tag{17}$$

It is seen that only azimuthally symmetrical E-modes ($n = 0$) contribute to the longitudinal impedance. Let us restrict ourselves to frequencies below the chamber cut-off, i.e. $\omega < \omega_c = \mu_{01} c/b$. This restriction is only for technical simplicity since in this case we can easily integrate over z in Eq.(17) without worrying about conditions at $z \to \pm\infty$. After carrying out this integration we get

$$Z(\omega) = -\frac{1}{q} \sum_{m=1}^{\infty} \frac{1}{\Gamma_{0m}} \left[b_{0m}(\Gamma_{0m} + ik) - a_{0m}(\Gamma_{0m} - ik) \right] \ .$$

Substituting a_{0m} and b_{0m} from Eq.(14) leads to

$$Z(\omega) = -i \frac{k}{q\varepsilon_0 b} \left[P_r + \frac{1}{c} M_\varphi \right] \sum_{m=1}^{\infty} \frac{\mu_{0m} J_1(\mu_{0m})}{\alpha_{0m}^2} \ ,$$

and, since $\alpha_{0m}^2 = \pi \mu_{0m}^2 J_1^2(\mu_{0m})$ (see Eq.(15)), to

$$Z(\omega) = -i \frac{k}{\pi q b} \left[\frac{P_r}{\varepsilon_0} + Z_0 M_\varphi \right] \sum_{m=1}^{\infty} \frac{1}{\mu_{0m} J_1(\mu_{0m})} \ .$$

The series in the RHS of this equation can be analytically summed up [9] and the sum is equal to 1/2. Taking into account the expressions (5) of the effective moments and Eqs.(2) with $n = 0$ for the beam fields, we obtain for an elliptic hole

$$Z(\omega) = -i \frac{Z_0 \omega}{4\pi^2 c b} \left(\alpha_e + \alpha_{m\parallel} \sin^2 \alpha + \alpha_{m\perp} \cos^2 \alpha \right) \ , \qquad (18)$$

where the polarizabilities are given by Eqs.(4). For the case of a circular hole we use the polarizabilities (6) and get a fairly simple expression for the longitudinal impedance:

$$Z(\omega) = -i \frac{Z_0}{6\pi^2} \frac{\omega h^3}{c b^2} \ , \qquad (19)$$

which shows an inductive contribution of the hole.[2] Respectively, the usually quoted quantity, so called reduced impedance, is

$$\frac{Z(n\omega_0)}{n} = -i \frac{Z_0}{6\pi^2} \frac{h^3}{R b^2} \ , \qquad (20)$$

[2] It will be recalled that we use $\exp(-i\omega t)$ time-dependence.

where $n = kR = \omega R/c$ is a harmonic number, $\omega_0 = c/R$ is the revolution frequency and R is the machine radius.

For a narrow longitudinal slot whose width w is much less than length l we use Eqs.(7) instead of (6) and obtain in the same way $Z(\omega) = 0$, i.e. the longitudinal impedance of a very narrow slot vanishes to a first-order approximation of our approach. To get a nonvanishing expression, it seems necessary to take into account next-to-leading terms in (7), i.e. while expanding Eqs.(4) with $\varepsilon \to 1$, and contributions from quadrupole term, see Eq.(13).

It should be mentioned that we have obtained the same results, Eqs.(18) and (19), also with a slightly different approach. We have considered the excitation of a hole by a single perturbation mode instead of a point charge and used the corresponding definition [10] of the longitudinal impedance.

3 The transverse impedance

With the notations introduced above the dipole transverse impedance is defined by

$$\vec{Z}_\perp(\omega, \vec{r}, \vec{a}) = -\frac{i}{qa}\int_{-\infty}^{\infty} dz e^{-ikz} \quad [\quad \vec{a}_r(E_r - Z_0 H_\varphi)(r, \varphi, z; \omega) \qquad (21)$$
$$+ \quad \vec{a}_\varphi(E_\varphi + Z_0 H_r)(r, \varphi, z; \omega) \ ] \ ,$$

where the Fourier-components of the e.m. fields have to be taken along the path of a test particle, which has the transverse offset $\vec{r} = (r, \varphi)$, and the coordinate unit vectors $\vec{a}_r$, $\vec{a}_\varphi$ correspond to this point. The limit of $r \to 0$, $a \to 0$ is usually assumed in Eq.(21). It is clear that both the E- and H-eigenmodes contribute to the integral. Note also that in an axisymmetric structure $Z_\varphi = 0$ but the hole breaks this symmetry. After some calculations which are similar to those for the longitudinal case we obtain for an elliptic hole with the major axis oriented along the longitudinal direction ($\alpha = 0$)

$$\vec{Z}_\perp(\omega, \vec{r}, \vec{a}) = -i\frac{Z_0}{\pi^2 b^4} \quad [\ \vec{a}_r \cos(\varphi - \varphi_0) - \vec{a}_\varphi \sin(\varphi - \varphi_0) \ ] \qquad (22)$$
$$\cdot \quad (\alpha_{m\perp}\Sigma_1 \cos\varphi_0 + \alpha_e \Sigma_2 \sin\varphi_0) \ .$$

One can easily get convinced that the vector sum in the square brackets is just the unit vector in the direction of the hole. We shall denote it as $\vec{a}_r(\varphi_0)$.

So, there is no dependence on φ, i.e. on the position of the test particle. The same holds with respect to $a = |\vec{a}|$, the value of the initial particle offset. Two series enter Eq.(22). The first one can be analytically summed [9]

$$\Sigma_1 = \sum_{m=1}^{\infty} \frac{\mu'_{1m}}{(\mu'^2_{1m} - 1)J_1(\mu'_{1m})} = 1 \ ,$$

where $J'_1(\mu'_{1m}) = 0$. But the second series,

$$\Sigma_2 = \sum_{m=1}^{\infty} \frac{1}{J_0(\mu_{1m})} \ ,$$

seems divergent. To be more precise, it is a conditionally-convergent series, i.e. the sum depends on the prescription of summation. For instance, taking the prescription $\Sigma_2 \equiv \lim_{n\to\infty}(\Sigma_2^n + \Sigma_2^{n+1})/2$, where $\Sigma_2^n \equiv \sum_{m=1}^{n}(J_0(\mu_{1m})^{-1}$ is the nth partial sum, leads to $\Sigma_2 = -1$. However, we can fix this problem in another way. Instead of inventing the prescription let us look at the symmetry of the problem.[3] It is rather obvious that $|\vec{Z}(\varphi_0)| = |\vec{Z}(-\varphi_0)|$. Hence, the function in the round brackets in Eq.(22) has to be an even function of φ_0. So, Σ_2 must be equal to zero! Therefore, the correct prescription, which respects the problem symmetry, has to give $\Sigma_2 = 0$. As a result, the transverse impedance of a single small elliptic hole is

$$\vec{Z}_\perp(\omega) = -iZ_0 \frac{\alpha_{m\perp}}{\pi^2 b^4} \vec{a}_r(\varphi_0) \cos\varphi_0 \ . \tag{23}$$

This means that the deflecting force is directed to (or opposite to) the hole and its value depends on the azimuthal angle φ_0 between the beam-offset vector and the direction to the hole.

For two particular cases we can obtain from Eq.(23): the transverse impedance of a circular hole is

$$\vec{Z}_\perp(\omega) = -iZ_0 \frac{4h^3}{3\pi^2 b^4} \vec{a}_r(\varphi_0) \cos\varphi_0 \tag{24}$$

and that of a narrow slot is

$$\vec{Z}_\perp(\omega) = -iZ_0 \frac{lw^2}{24\pi b^4} \vec{a}_r(\varphi_0) \cos\varphi_0 \ . \tag{25}$$

[3]The author is indebted to Dr. G. Dôme whose remark on this symmetry gave a hint on the solution of the problem.

If we consider now two opposite holes in one cross section and assume an additivity of the low-frequency impedance, that is customary, the result will differ from (24) or (25) by a factor 2. But when we consider M ($M \geq 3$) holes uniformly spaced in one cross section, the resulting impedance is

$$\vec{Z}_\perp(\omega) = -iZ_0 \frac{2h^3}{3\pi^2 b^4} M \vec{a}_1 \ , \tag{26}$$

and for M slots

$$\vec{Z}_\perp(\omega) = -iZ_0 \frac{lw^2}{48\pi b^4} M \vec{a}_1 \ , \tag{27}$$

where $\vec{a}_1 = \vec{a}/|\vec{a}|$ is the unit vector in the direction of the beam transverse offset. It is seen that

- the deflecting force is now directed along the beam displacement, i.e. some restoration of the axial symmetry occurs;
- the maximum value of $|\vec{Z}_\perp|$ is only $M/2$ times larger than that for $M = 1$.

4 Many holes and a few estimates

4.1 Many holes

The results of previous Section concern a single hole or a few holes in one chamber cross section. But usually there are a lot of holes along the beam path. Let their longitudinal spacing be d. When $d \gg h$ we can use the Bethe theory as in the case of a single hole, but the field pattern is now a result of superposition from all holes. Moreover, different holes are excited with different phases. So, some kind of interference seems to be present. But we will argue that one can obtain reasonable estimates while disregarding these considerations.

First of all, we deal with low frequencies, below cut-off. It is the common wisdom that small discontinuities contribute additively to the coupling impedance in this range. Indeed, an expected condition of a strong interference has to be of the form $kd = 2\pi m$, m is an integer. It assumes frequencies above the chamber cut-off when $d \leq b$. On the other hand, if $d \gg b$, the excited waveguide modes from one hole do not reach the next one. It follows from the fact that for $k = 2\pi m/d$ the attenuation length is of the order of $L = \Gamma_{01}^{-1} \simeq b/\mu_{01} \ll d$. So, we shall assume an additivity at low frequencies for further estimates. But, in fact, this problem remains to be examined more carefully. We also refer the reader to the paper by M. Sands [5] which contains some consideration on the subject.

With this assumption let us first compare the longitudinal impedance produced by pumping holes in the shield of a small cavity with that of the unshielded cavity. Let this cavity have a depth Δ and length g, with $\Delta \ll b$ and $g \ll b$, and let the ratio κ of the whole area of holes, $N\pi h^2$, to the cavity area seen by the beam, $2\pi bg$, be fixed from vacuum requirements. The low-frequency longitudinal impedance of the open cavity would be approximately [10]

$$Z_c(\omega) \simeq -iZ_0 \frac{\omega}{2\pi c} \frac{\Delta g}{b}$$

and that of $N = 2\kappa bg/h^2$ holes from Eq.(19) is

$$Z_h(\omega) = -iZ_0 \frac{\omega}{2\pi c} \frac{2hg}{3\pi b} \kappa \ .$$

Hence, shielding reduces the impedance by a factor

$$\frac{Z_h}{Z_c} \simeq \frac{2h}{3\pi\Delta} \kappa \ ,$$

which is small compared to unity since usually $\kappa \leq 0.2$ and $h \ll \Delta$.

4.2 Estimates

First, consider the impedances produced by the pumping slots in so called liners in the UNK [2] vacuum chamber, which are the e.m. shields. Approximately $N = 3260$ vacuum boxes with bellows will be shielded by these liners and every liner has $M = 26$ narrow pumping slots with width $w = 0.6$ cm and length $l = 6$ cm. For our estimates we shall take the chamber radius $b = 3.5$ cm and the machine one $R = 3306$ m. To be rigorous, for this case we are nearly beyond the framework of our approach, since $l > b$, and the estimates obtained are rough. Moreover, for the longitudinal impedance of the slots we show here only an upper limit, namely, the contribution from magnetic term, which is equal to and cancels, to a first-order approximation, the electric one. It seems naturally to expect that a second-order result will be smaller. Our low-frequency impedance estimates are shown in Table 1. For a single slot $Z_\perp$ stands for the maximum value of $|\vec{Z}_\perp|$, i.e. with $\cos\varphi_0 = 1$, see Eq.(25).

Table 1: Impedance Estimates for the UNK Slots

	$\|Z/n\|$, Ohm	$Z_\perp$, Ohm/m
One slot	$< 7 \cdot 10^{-7}$	7.2
One liner	$< 2 \cdot 10^{-5}$	93.6
Total	< 0.06	$3.1 \cdot 10^5$

These values of the coupling impedances are not dangerous.

In the LHC design [3] it is supposed to shield the vacuum chamber which will be at 1.9 K by an internal thermal screen. It will contain approximately $N = 10^7$ vacuum pumping holes with radius $h = 1 \div 2$ mm. The longitudinal spacing is $d = 1$ cm and the machine radius is $R = 4243$ m. Hence, there will be nearly $M = 4$ holes in a chamber cross section. We shall take the mean radius of the thermal screen $b = 1.5$ cm for our estimate. The figures are shown in Table 2.

Table 2: Impedances of the LHC Holes

	$\|Z/n\|$, Ohm		$Z_\perp$, Ohm/m	
	$h = 1$ mm	$h = 2$ mm	$h = 1$ mm	$h = 2$ mm
One hole	$6.7 \cdot 10^{-9}$	$5.3 \cdot 10^{-8}$	1	8.05
One cross section	$2.7 \cdot 10^{-8}$	$2.1 \cdot 10^{-7}$	2	16.1
Total	0.07	0.53	$5 \cdot 10^6$	$4 \cdot 10^7$

The values of the longitudinal and especially transverse total impedance seem unacceptably large even in the case of smaller holes. So, we conclude that some modifications of this thermal-screen construction are necessary.

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Note added

After this work was completed, the author has been informed[4] of a recent paper [11] treating the same subject. Using the 3 dimensional code T3 in MAFIA [12] the authors of paper [11] have estimated the contribution of a single hole to the longitudinal wake field. Fitting the numerical results leads to $|Z/n| \sim \omega_0 Z_0 h^3/(cb^2)$, in our notations, and the numerical factor approximately coincides with that of Eq.(20).

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Appendix

On the real part of the longitudinal impedance

We have obtained the purely reactive longitudinal impedance of a single small round hole, Eq.(19). It will be recalled that we have considered the perfectly conducting walls and frequencies below cut-off. But the beam energy is lost to excite the hole. We can estimate the real part of the impedance in an indirect way, calculating the energy radiated by the hole. This evaluation is essentially the same one as in the paper by M. Sands [5].

Let a bunch with normalized longitudinal charge distribution $\lambda(s)$ move along the chamber axis with velocity $v = c$. In the case of a point charge $\lambda(s) = \delta(s)$. The total energy radiated in vacuum by the effective dipoles $\vec{M}(t)$ and $\vec{P}(t)$ is

$$\Delta U = \int_{-\infty}^{\infty} dt W(t) = \frac{Z_0}{6\pi c^2} \int_{-\infty}^{\infty} dt \left[\frac{1}{c^2} \left(\frac{d^2}{dt^2} \vec{M}(t) \right)^2 + \left(\frac{d^2}{dt^2} \vec{P}(t) \right)^2 \right] .$$

Since $E_r(z - ct) = Z_0 H_\varphi(z - ct) = q\lambda(z - ct)/(2\pi b \varepsilon_0)$ we have for a round hole

$$\vec{P}(t) = -\frac{qh^3}{3\pi b} \lambda(z - ct)$$

and

$$\vec{M}(t) = -\frac{2qh^3 c}{3\pi b} \lambda(z - ct) .$$

Then we rewrite ΔU in the ω-domain:

$$\Delta U = \frac{5Z_0 q^2 h^6}{108\pi^4 b^2 c^4} \int_{-\infty}^{\infty} d\omega \omega^4 |\lambda(\omega)|^2 \ ,$$

where $\lambda(\omega) = \int ds \exp(i\omega s/c)\lambda(s)$ is the bunch spectrum ($\lambda(\omega) = 1$ for a point charge).

On the other hand, $\Delta U = kq^2$, where the loss factor k is

$$k = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega Re\, Z(\omega) |\lambda(\omega)|^2 \ .$$

So,

$$\Delta U = \frac{1}{2\pi} q^2 \int_{-\infty}^{\infty} d\omega Re\, Z(\omega) |\lambda(\omega)|^2 \ .$$

Comparing two expressions of ΔU and taking into account that the integrands are positively defined we get for a round hole

$$Re\, Z(\omega) = \frac{5Z_0}{54\pi^3} \left(\frac{\omega h}{c}\right)^4 \frac{h^2}{b^2} \ .$$

The calculation of the energy loss of a Gaussian bunch with this expression reproduces the result by M. Sands.

One can easily recognize that in the framework of our assumptions, i.e. with $\omega h/c \ll 1$ and $h \ll b$, the inequality

$$Re\, Z \ll |Im\, Z|$$

holds.

The same approach gives in the case of a narrow slot

$$Re\, Z(\omega) = \frac{Z_0}{6912\pi} \left(\frac{\omega w}{c}\right)^4 \frac{l^2}{b^2} \ .$$