TRANSVERSE RESISTIVE WALL IMPEDANCE FOR MULTI-LAYER ROUND CHAMBERS

Alexey Burov and Valeri Lebedev, FNAL

Abstract

The resistive wall impedance is usually calculated assuming the skin depth being much smaller than the chamber thickness. This approximation is not always correct. In particular, it is not valid when the revolution frequency is very low (as in VLHC [1]), or the surface is coated by a thin conductive layer (as for extraction kickers [2]), or for the coherent effects in the closed orbit motion [3]. A method of analytical calculation of the transverse impedance is developed here for multi-layer vacuum chambers and applied to an arbitrary two-layer structure.

1 METHOD OF CALCULATIONS

Calculations of the electromagnetic fields excited by the beam dipole motion are simplified when the wave length of the beam transverse oscillations is much longer than the aperture, $c/\omega \gg a$. This condition is always satisfied, if the finite skin depth plays a role. Charges and currents excited on the chamber surface feel only local beam offset; thus, they feel the beam as moving in parallel to the chamber axis with small transverse oscillations at a given frequency ω.

This dipole motion of the beam can be presented as a superposition of oscillating electric and magnetic dipoles: one is due to the beam charge and another due to its current [1]. The chamber response on the electric dipole is just the electrostatic screening. For this response, the transverse electric field $\vec{E} = -\nabla\Phi$ can be found from the scalar potential Φ excited by the beam offset $x_0 \exp(-i\omega t)$ in a round vacuum chamber:

$$\Phi(r,\theta) = \frac{A_0}{\beta}\left(\frac{a}{r} - \frac{r}{a}\right)\cos\theta\, e^{-i\omega t}\,, \quad A_0 = \frac{2Ix_0}{ca} \tag{1}$$

where I is the beam current, r, θ are the polar coordinates, and a is the chamber radius.

The oscillating magnetic dipole gives rise to the transverse magnetic field. Due to a fact that the longitudinal wave length is much larger than the aperture, all the fields can be expressed through the longitudinal vector potential A_z.

Because of the chamber axial symmetry, the vector potential can be written as $A_z = A(r)\cos\theta\, e^{-i\omega t}$; thus, only its radial factor $A(r)$ has to be found. For round chambers, this radial factor satisfies Bessel equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) - \frac{A}{r^2} - \kappa^2 A = 0\,, \tag{2}$$

where

$$\kappa^2 = -\frac{\omega^2\mu\varepsilon}{c^2}\,, \quad \varepsilon = 1 + \frac{4\pi i\sigma}{\omega} \tag{3}$$

with μ, ε as the magnetic and dielectric constants of the medium, and σ as its conductivity.

For a multi-layer problem, Eq. 2 has to be solved with the following boundary conditions:

- The vector potential has a singularity at $r = 0$ due to the beam magnetic dipole: $\lim_{r\to 0} Ar = 2Ix_0/c$.
- At any boundary, A and $\mu^{-1}dA/dr$ are continuous.
- The vector potential vanishes at infinity.

Inside the vacuum chamber $\varepsilon = \mu = 1$, and the vector potential writes as

$$A(r) = A_0(a/r - Gr/a)\,, \tag{4}$$

where the constant G has to be found from the boundary conditions.

Inside the metal $4\pi\sigma \gg \omega$, and the vector potential satisfies the Bessel equation (2) with the wave vector

$$\kappa^2 = -4\pi i\sigma\mu\omega/c^2 \equiv -2i/\delta^2\,, \tag{5}$$

where $\delta^2 = c^2/(2\pi\sigma\mu\omega)$ is the skin-depth.

A solution of the Bessel equation can be presented as a superposition of its two basis functions, which can be taken as arbitrary combinations of the modified Bessel functions. It is convenient to take this basis as a generalization of $\sinh(\kappa(r-a))$ and $\cosh(\kappa(r-a))$, which compose a fundamental pair in case $\kappa a \gg 1$. In other words, it is convenient to choose such basis solutions $\mathrm{shb}(\kappa r)$ and $\mathrm{chb}(\kappa r)$ (“sine hyperbolic Bessel” and “cosine hyperbolic Bessel”) that

$$\left.\begin{matrix} \mathrm{chb}(\kappa r) = 1 & \mathrm{chb}'(\kappa r) = 0 \\ \mathrm{shb}(\kappa r) = 0 & \mathrm{shb}'(\kappa r) = 1 \end{matrix}\right|_{r=a+0}\,, \tag{6}$$

with a symbol $'$ staying for a derivative over the argument κr.

In terms of the modified Bessel functions $\mathrm{I}_1(\kappa r)$, $\mathrm{K}_1(\kappa r)$, the required combinations write:

$$\begin{aligned} \mathrm{shb}(\kappa r) &\equiv \kappa a[\mathrm{I}_1(\kappa r)\mathrm{K}_1(\kappa a) - \mathrm{I}_1(\kappa a)\mathrm{K}_1(\kappa r)]\,; \\ \mathrm{chb}(\kappa r) &\equiv \kappa a[\mathrm{I}_1'(\kappa a)\mathrm{K}_1(\kappa r) - \mathrm{I}_1(\kappa r)\mathrm{K}_1'(\kappa a)]\,. \end{aligned} \tag{7}$$

Here we used a property of the Bessel’s Wronskian: $\mathrm{I}_1'(\kappa a)\mathrm{K}_1(\kappa a) - \mathrm{I}_1(\kappa a)\mathrm{K}_1'(\kappa a) = 1/(\kappa a)$. By definition, these hyperbolic Bessel functions depend on two arguments, κa and κr. However, the first argument is always taken at the inner radius of the considered layer, so it may be safely omitted. For high value of the arguments, $\kappa a \gg 1$, (skin depth $\ll$ radii) the introduced hyperbolic

Bessel functions are reduced to the conventional hyperbolic functions:

$$\begin{aligned} \mathrm{shb}(\kappa r) &\to \sinh(\kappa(r-a))\,; \\ \mathrm{chb}(\kappa r) &\to \cosh(\kappa(r-a))\,. \end{aligned} \qquad (8)$$

Note that inside metal $\kappa \propto 1 \pm i$; thus, the used Bessel functions of complex arguments can be expressed in terms of Kelvin functions ber, bei, ker, kei of real arguments. For a non-conductor, the basis functions reduce to

$$\mathrm{shb}(\kappa r) = \frac{\kappa r}{2} - \frac{\kappa a^2}{2r}\,, \quad \mathrm{chb}(\kappa r) = \frac{r}{2a} + \frac{a}{2r}\,. \qquad (9)$$

The solution of the Bessel equation can be presented as

$$A/A_0 = C\,\mathrm{chb}(\kappa r) + S\,\mathrm{shb}(\kappa r)\,. \qquad (10)$$

For the outermost space, the solution depends on whether it is filled by a metal (with $\kappa = \kappa_o$, the subscript $_o$ stays for 'outermost') or non-conductor:

$$\frac{A}{A_0} \propto \left\{ \begin{array}{ll} 1/r & \text{non-conductor} \\ \mathrm{K}_1(\kappa_o r)\,, & \text{conductor} \end{array} \right. . \qquad (11)$$

As a result of the consequent solution of the boundary equations, the inner constant G in Eq. (4) can be found. After that, the transverse impedance per unit length is expressed as

$$Z_\perp = -i\frac{E_x - \beta H_y}{I x_0} = -i\frac{Z_0\beta}{2\pi a^2}\left(-G + \frac{1}{\beta^2}\right)\,, \qquad (12)$$

where $\beta = v/c$ and the term $\propto 1/\beta^2$ comes from a contribution of the electric dipole. Note that the transverse impedance is defined here in A. Chao's convention [5]. This can also be presented as

$$Z_\perp = -i\frac{Z_0\beta(1-G)}{2\pi a^2} - i\frac{Z_0}{2\pi a^2\beta\gamma^2} \equiv Z_\perp^\sigma + Z_\perp^\infty\,. \qquad (13)$$

For infinite conductivity $G = 1$, so the first term $Z_\perp^\sigma \propto 1 - G$ is a resistive wall impedance. The second term $Z_\perp^\infty \propto \beta^{-1}\gamma^{-2}$ describes image charges of the perfectly conducting wall; it vanishes in the relativistic limit.

2 TWO-LAYER CHAMBER

In this section, the impedance is found for an arbitrary two-layer chamber. It is convenient to use subscripts $1, 2$ for values related to the first or second layer. The inner layer with the thickness $d = a_2 - a_1$ is a metal with the wave vector $\kappa_1 = \sqrt{-4\pi i\sigma_1\mu_1\omega/c^2}$ and the medium parameter $\tilde\kappa_1 = \kappa_1/\mu_1$. The second layer is not bounded from the outside and characterized by the medium parameter $\tilde\kappa_2 = \kappa_2/\mu_2$ if it is conductive, otherwise $\tilde\kappa_2 = 1/(a_2\mu_2)$.

The continuity conditions for A and $\mu^{-1}dA/dr$ follow:

$$\begin{array}{c} 1 - G = C \\ -1 - G = S\tilde\kappa_{10} \\ \\ Cc_1 + Ss_1 = G_o \\ Cc_1' + Ss_1' = -G_o\tilde\kappa_{21} \end{array} . \qquad (14)$$

Here $c_1 = \mathrm{chb}(\kappa_1 a_2)\,,\ s_1 = \mathrm{shb}(\kappa_1 a_2)\,,\ c_1' = \mathrm{chb}'(\kappa_1 a_2)\,,\ s_1' = \mathrm{shb}(\kappa_1 a_2)$; the parameters $\tilde\kappa_{10} = \tilde\kappa_1 a_1$ and $\tilde\kappa_{21} = \tilde\kappa_2/\tilde\kappa_1$ reflect relative properties of adjacent media, and the constant G_o describes vector potential in the outer layer.

Making a ratio from the first pair of the boundary equations leads to an expression of the impedance factor $1 - G$ in terms of the amplitude ratio $T = S/C$:

$$1 - G = 2/(1 - \tilde\kappa_{10}T)\,. \qquad (15)$$

Similarly, a pair of equations at the outer boundary leads to

$$T = -(\tilde\kappa_{21}c_1 + c_1')/(s_1' + \tilde\kappa_{21}s_1)\,. \qquad (16)$$

Moving back from Eq. (16) to Eq. (15) and then to Eq. (12), the impedance is found:

$$Z_\perp^\sigma = -i\frac{Z_0\beta}{\pi a^2}\frac{s_1' + \tilde\kappa_{21}s_1}{s_1' + \tilde\kappa_{21}\tilde\kappa_{10}c_1 + \tilde\kappa_{21}s_1 + \tilde\kappa_{10}c_1'} \qquad (17)$$

for an arbitrary two-layer chamber, with the only assumption that the longitudinal wave length is long: $v/\omega \gg a_1, a_2$. The inner layer is a metal with an arbitrary skin depth, the outer medium is not bounded outside, and it can be either conductive or not.

An example of the impedance behavior as a function of $ad/\delta^2 \propto \omega$ is shown in Fig 1. The inner metal layer is taken with the thickness-to-radius ratio $d/a = 0.05$. The outer medium is either vacuum or a non-conductive magnetic.

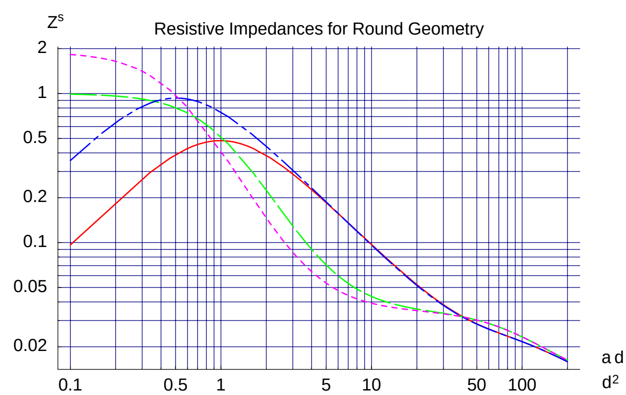

Figure 1: Resistive impedance in units of $Z_0/(2\pi a^2)$ as a function of $ad/\delta^2 \propto \omega$. Solid red and dashed green lines give its real and negated imaginary parts for $d/a = 0.05\,, \mu_1 = 1$ and vacuum outside. The dot-dashed blue and dotted magenta lines are same functions when the outside medium is a non-conductive magnetic with $\mu_2 = 500$. Calculations are performed with *Mathematica* [6].

Three different regions can be distinguished here:

- High frequencies, $\delta < d$: the conventional result $Z_\perp^\sigma \propto 1/\sqrt{\omega}\,,\ \mathrm{Re}Z_\perp^\sigma = -\mathrm{Im}Z_\perp^\sigma$ is valid.
- Moderate frequencies, $d < \delta < \sqrt{ad}$: the impedance goes as $Z_\perp^\sigma \propto 1/\omega\,,\ \mathrm{Re}Z_\perp^\sigma \gg |\mathrm{Im}Z_\perp^\sigma|$.

- Low frequencies, $\delta > \sqrt{ad}$: the real part of impedance linearly goes to 0, while the imaginary part to a non-zero constant. A non-zero value of the imaginary part at $\omega = 0$ corresponds to dipole fields of image charges (and currents, with magnetic outside) driven by the beam offset.

The real part of impedance reaches maximum in the boundary of low and intermediate frequencies, $\delta \simeq \sqrt{ad}$. Apart from the low-frequency region, the magnetic permeability of the outside medium μ does no influence the impedance. At low frequencies, the magnetic doubles the imaginary part of impedance and gives a factor of 4 to its small real part.

Except the extremely low frequencies, with $\delta \geq a$, where the impedance is almost pure imaginary, the Bessel functions can be substituted by their asymptotics, the hyperbolic functions. This allows to simplify the result (17) as

$$Z_\perp^\sigma = -i\frac{Z_0\beta}{\pi a^2}\frac{1+\tilde{\kappa}_{21}t_1}{1+\tilde{\kappa}_{21}\tilde{\kappa}_{10}+(\tilde{\kappa}_{21}+\tilde{\kappa}_{10})t_1}\,, \qquad (18)$$

with $t_1 = \tanh(\kappa_1 d)$. It can be deduced from here that the impedance is independent of the outer layer when $\sqrt{\sigma_2/\mu_2} \ll \sqrt{\sigma_1/\mu_1}\tanh(\kappa_1 d)$.

The impedance may be specified for many particular cases. For example, when the outer medium is vacuum, and both the skin depth and the chamber thickness are small in comparison with the radius, the impedance (18) writes as

$$Z_\perp^\sigma = -i\frac{Z_0\beta}{\pi a^2}\frac{1+\frac{\mu}{\kappa a}\tanh(\kappa d)}{2+\left(\frac{\mu}{\kappa a}+\frac{\kappa a}{\mu}\right)\tanh(\kappa d)}\,. \qquad (19)$$

If the outer layer is a non-conductive magnetic with $\mu_2 \gg 1$, and the inner layer is a non-magnetic metal, then

$$Z_\perp^\sigma = -i\frac{Z_0\beta}{\pi a^2}\frac{1}{1+\kappa a\tanh(\kappa d)}\,. \qquad (20)$$

This result shows the impedance saturation with the outside permeability.

The transverse impedance for a thin chamber with vacuum outside is presented (without a derivation) in Ref. [4] (see Eq. (6.122) at p. 170). That result is identical to our Eq. (19) at $\kappa d \ll 1$ and $\mu = 1$. However, when $\mu \neq 1$, the results are significantly different; we leave it for the readers to decide what is correct.

3 THREE AND MORE LAYERS

The described method can be directly applied for an arbitrary number of layers n. To find the impedance (12), one has to solve a consequence of n simple recursive linear equations, expressing one unknown value - the amplitude ratio for an inner layer - through this value for an adjacent outer layer, which comes at the previous step. Note that the amplitude ratio in the layer $n-1$ is known - it is given by Eq. (16):

$$T_{n-1} = -\frac{c'_{n-1}+\tilde{\kappa}_{n,n-1}c_{n-1}}{s'_{n-1}+\tilde{\kappa}_{n,n-1}s_{n-1}}\,. \qquad (21)$$

A transfer from a layer m to a layer $m-1$, with $2 \leq m \leq n-1$, is found as

$$T_{m-1} = -\frac{c'_{m-1}-T_m\tilde{\kappa}_{m,m-1}c_{m-1}}{s'_{m-1}-T_m\tilde{\kappa}_{m,m-1}s_{m-1}}\,. \qquad (22)$$

Finally, a transfer from the layer 1 to the 'layer 0', or inner part of the vacuum chamber, is given by Eq. (15), assuming $T = T_1$ there. In this section, all the notations are just slightly generalized from ones of the previous section: $c_{m-1} = \mathrm{chb}(\kappa a_m)\,,\ \tilde{\kappa}_{m,m-1} = \tilde{\kappa}_m/\tilde{\kappa}_{m-1}$, etc. Remember that the used basis Bessel solutions $\mathrm{chb}(\kappa r), \mathrm{shb}(\kappa r)$ depend also on the inner radius of the corresponding layer, according to Eqs. (6, 7) with a as this radius. In other words,

$$c_{m-1} = q_+(\mathrm{I}'_1(q_+)\mathrm{K}_1(q_-) - \mathrm{I}_1(q_-)\mathrm{K}'_1(q_+))\,, \qquad (23)$$

with $q_+ = \kappa_{m-1}a_m\,,\ q_- = \kappa_{m-1}a_{m-1}$ and similar for s_{m-1}. Substituting the presented formulae one into another, the impedance is calculated analytically for any number of layers.

4 CONCLUSIONS

An effective method is found for analytical calculation of the transverse resistive wall impedance for multi-layer round vacuum chambers. A compact formula is derived for an arbitrary two-layer radial structure.

An opposite limit to the round shape of the vacuum chamber is flat one, when it can be approximated by a pair of parallel multi-layer plates. This problem is solved in our next paper [7].

5 REFERENCES

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