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A Mechanism of Longitudinal Single-Bunch Instability in Storage Rings

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ABSTRACT

A new type of longitudinal single-bunch instability in storage rings is found. This instability is resulted from an interaction of two coherent synchrotron motions with different amplitudes. The frequency spread of incoherent synchrotron motion in a bunch generated by the potential-well distortion plays an essential role in this instability. The system becomes unstable when synchrotron frequencies of two different amplitudes degenerate. In an extreme case with the pure-resistive (δ-function) wake potential, it is shown that the system is always unstable, *i.e.*, the threshold intensity is zero.

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The longitudinal single-bunch collective motion in a storage ring is usually described by the Vlasov equation [1]

$$-\frac{\partial f}{\partial s} = -p\frac{\partial f}{\partial q} + (q - V(q,s))\frac{\partial f}{\partial p} \qquad (1)$$

for the distribution function $f = f(p,q,s)$ in the longitudinal phase space. The independent variables are $p \equiv (E - E_0)/E_0\sigma_\epsilon$ (relative energy deviation), $q \equiv z/\sigma_z$ (relative longitudinal position), and $s \equiv \omega_s t$ (phase of the synchrotron motion). We have introduced E_0 as the nominal beam energy, σ_ϵ the natural energy spread, σ_z the natural bunch length, and ω_s the unperturbed angular frequency of the synchrotron motion. The positive q corresponds to the front of the bunch. Here the external rf-field is assumed to be linear in the position q. The charge of the bunch induces the longitudinal force $-V(q)$ with the longitudinal wake function (Green's function) $W(q)$ as

$$V(q,s) = k\int_{-\infty}^{+\infty} \rho(q',s)W(q'-q)dq' , \qquad (2)$$

where $\rho(q,s) = \int_{-\infty}^{+\infty} f(p,q,s)dp$ is the longitudinal density of particles, and normalized as $\int \rho(q,s)dq = 1$. The parameter k represents the beam intensity:

$$k = \frac{Ne}{2\pi\nu_s\sigma_\epsilon}\left(\frac{e}{E_0}\right) , \qquad (3)$$

where N is the number of the particles in the bunch and ν_s is the unperturbed synchrotron tune. We only consider an ultra-relativistic case, which mean $W(q) = 0$ for $q < 0$.

The equilibrium solution of Eq. (1) is written as

$$f_0(p,q) = g\left(H(p,q)\right) , \qquad (4)$$

where $H(p,q)$ is Hamiltonian for the single-particle motion in the bunch:

$$H(p,q) = \frac{p^2}{2} + \frac{q^2}{2} - \int_0^q V_0(q')dq' . \qquad (5)$$

The equation of motion of a particle with this Hamiltonian is

$$\frac{dp}{ds} = \frac{\partial H}{\partial q} , \qquad \frac{dq}{ds} = -\frac{\partial H}{\partial p} . \qquad (6).$$

The wake voltage V_0 in Eq. (5) is determined by the density $\rho_0(q) = \int f_0(p,q)dp$ using Eq. (2) self-consistently [2]:

$$V_0(q) = k\int_{-\infty}^{+\infty} \rho_0(q')W(q'-q)dq' . \qquad (7)$$

The deformation of the distribution f_0 and the voltage V_0 by the intensity through the wake potential is called "potential-well distortion", which will play an essential role in the longitudinal single-bunch instability. The actual form of the function g in (4) is not unique, but in the case of an electron-storage ring, the function g must be Gaussian in p-direction, $g(H) \propto \exp(-H)$, to be consistent with the damping and diffusion caused by the synchrotron radiation.

The equilibrium solution (4) may exist in most cases with arbitrary intensity k, but this does not guarantee the stability of the solution. The stability of the stationary solution (4) is examined by a linear perturbation. We expand f around the stationary distribution f_0 as $f(p,q,s) = f_0(p,q) + f_1(p,q,s)$, and take the first order terms of f_1 in Eq. (1), then obtain

$$-\frac{\partial f_1}{\partial s} = -p\frac{\partial f_1}{\partial q} + (q - V_0(q))\frac{\partial f_1}{\partial p} - V_1(q,s)\frac{\partial f_0}{\partial p} , \qquad (8)$$

where V_1 is the wake voltage induced by f_1 with Eq. (2).

One should not neglect or approximate the term of potential-well distortion $-V_0(q)\frac{\partial f_1}{\partial p}$ in Eq. (8), not only because it is in the same order of the intensity k as another term $-V_1(q,s)\frac{\partial f_0}{\partial p}$. Once the potential-well term is neglected, the consistency of Eq. (8) will be lost, and it leads to unphysical results. For instance, since the wake potential is an internal force of the bunch, a simple sinusoidal motion of the centroid of the bunch, with the frequency ω_s, is never affected by the longitudinal wakefield. Therefore there always exists one trivial solution for Eq. (1) which corresponds to the motion of the centroid of the bunch. The solution is

$$f(p,q,s) = \Re f_0\left(p + ia\exp(is), q - a\exp(is)\right) , \qquad (9)$$

where a is an arbitrary amplitude of the motion of the centroid. Thus the first order deviation of (9) from f_0 for a small a

$$f_1 = \Re\left[a\exp(is)\left(-\frac{\partial f_0}{\partial q} + i\frac{\partial f_0}{\partial p}\right)\right] \qquad (10)$$

satisfies the first-order equation (8). If one modifies the potential-well term in Eq. (8), the centroid motion (10) becomes no longer its solution. Therefore by changing the potential-well term, one may get an unphysical mode of the motion of the bunch instead of the trivial but physical solution. Couplings of such unphysical modes may give incorrect information on the stability.

To solve the problem of the stability as exactly as possible, a formulation has been introduced for numerical computation [3]. This method uses action-angle variables (J,ϕ) to rewrite the Hamiltonian (5) as $H = H(J)$. Here we define these variables as

$$q \to \sqrt{2J}\cos\phi , \qquad p \to \sqrt{2J}\sin\phi , \qquad (11)$$

in the limit $k \to 0$. These variables rewrite Eq. (8) to

$$\begin{aligned} -\frac{\partial f_1}{\partial s} &= \omega(J)\frac{\partial f_1}{\partial \phi} - V_1(q,s)\frac{\partial f_0}{\partial p} \\ &= \omega(J)\frac{\partial f_1}{\partial \phi} - pV_1(q,s)g'(H(J)) , \end{aligned} \qquad (12)$$

where $\omega(J) = d\phi/ds = \partial H/\partial J$ is the angular frequency of the single-particle motion in the potential well. In Eq. (12), we have applied Eqs. (4) and (5). Then we expand f_1 in terms of an orthogonal base as

$$f_1 = \sum_{jm} m\omega_j\left(g'_j\Delta J_j\right)^{1/2} h_j(J)\exp(-i\mu s)(C_{jm}\cos m\phi + S_{jm}\sin m\phi) , \qquad (13)$$

where the function $h_j(J)$ is a step-like function and takes the value $1/\Delta J_j$ in the strip around the j-th mesh point $J = J_j$ with the thickness ΔJ_j, and zero outside. We have also used the factor $m\omega_j\left(g'_j\Delta J_j\right)^{1/2}$ with $g'_j \equiv g'(H(J_j))$ and $\omega_j \equiv \omega(J_j)$ for convention. Since Eq. (12) contains no derivative by J, we do not have to worry about the discontinuity of $h_j(J)$. After substituting (13) to (12), we multiply $\left(\Delta J_j/g'_j\right)^{1/2} h_j(J)\cos m\phi/\pi m\omega_j$ or $\left(\Delta J_j/g'_j\right)^{1/2} h_j(J)\sin m\phi/\pi m\omega_j$ on the both side of them, and integrate them over J and ϕ, then obtain

$$
\begin{aligned}
i\mu C_{jm} =& m\omega_j S_{jm} \\
i\mu S_{jm} =& -m\omega_j C_{jm} \\
& -k\sum_{j'm'} \frac{m'\omega_{j'}\left(g'_j\Delta J_j\right)^{1/2}\left(g'_{j'}\Delta J_{j'}\right)^{1/2}}{\pi m\omega_j} C_{j'm'} \int_0^\infty dJ h_j(J) \int_0^\infty dJ' h_{j'}(J') \\
& \times \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' p(J,\phi) W\left(q(J',\phi') - q(J,\phi)\right) \sin m\phi \cos m'\phi' \ .
\end{aligned} \tag{14}
$$

In the integral to get V_1, we have changed the variables from (p', q') to (J', ϕ') with $dp'dq' = dJ'd\phi'$. We have also assumed the functions $g'(H(J))$ and $\omega(J)$ are smooth enough in the width of $h_j(J)$, so that represented them by the central values g'_j and ω_j, respectively. Making use of the equation of motion $p = -\omega(J)\partial q/\partial\phi$, we can rewrite the integrand of (14) as

$$
p(J,\phi)W\left(q(J',\phi') - q(J,\phi)\right) = \omega(J)\frac{\partial}{\partial\phi}F\left(q(J',\phi') - q(J,\phi)\right) \ , \tag{15}
$$

where F is a primitive function of W, *i.e.*, $F'(q) = W(q)$. If we substitute (15) into (14) and integrate it by part by ϕ, the second equation of (14) becomes

$$
\begin{aligned}
i\mu S_{jm} =& -m\omega_j C_{jm} + \frac{k}{\pi}\sum_{j'm'} m'\omega_{j'}\left(g'_j\Delta J_j\right)^{1/2}\left(g'_{j'}\Delta J_{j'}\right)^{1/2} C_{j'm'} \\
& \times \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' F\left(q(J_{j'},\phi') - q(J_j,\phi)\right) \cos m\phi \cos m'\phi' \ .
\end{aligned} \tag{16}
$$

We have again assumed the smoothness of ω, q, and F in the strip $h_j(J)$ to evaluate their integrals with the values at J_j and $J_{j'}$. Combining (16) and the first equation

of (14), we get a linear equation for C_{jm}:

$$
-\mu^2 C_{jm} = -\sum_{j'm'} M_{jmj'm'} C_{j'm'} \ , \tag{17}
$$

with

$$
\begin{aligned}
M_{jmj'm'} =& m^2\omega_j^2\delta_{jj'}\delta_{mm'} - \frac{k}{\pi} m m'\omega_j\omega_{j'}\left(g'_j\Delta J_j\right)^{1/2}\left(g'_{j'}\Delta J_{j'}\right)^{1/2} \\
& \times \int_0^{2\pi}\int_0^{2\pi} \cos m\phi \cos m'\phi' F\left(q(J_{j'},\phi') - q(J_j,\phi)\right) d\phi d\phi' \ ,
\end{aligned} \tag{18}
$$

where $\delta_{jj'}$ is the Kronecker's delta. The system becomes unstable when the matrix M has a negative or a complex eigenvalue.

Once the wake potential $W(q)$ of a machine is given, we can examine the stability of the equilibrium (4) by solving the matrix (18) for given numbers of mesh points in J-direction and azimuthal modes. This method has been applied to several cases [4].

To proceed further in an analytical way, we apply this formulation to special forms of the wake potential: pure-capacitive: $W(q) = C\theta(q)$, pure-resistive: $W(q) = R\delta(q)$, and pure-inductive: $W(q) = -L\delta'(q)$ wakes. The function $\theta(q)$ is the step function defined as $\theta(q) = 1$ when $q > 0$ and zero otherwise. First in the cases of the pure-capacitive and the pure-inductive wakes, the matrix $M_{jmj'm'}$ becomes completely symmetric for exchange of (j, m) and (j', m') indices. In Eq. (18), it is easy to see that the matrix M becomes symmetric when F is an even function. In the case of the pure-capacitive wake, we can choose $F(q) = C(|q| + q)/2$. Its first term is even in q and the second term vanishes in the integral of (18), then M is symmetric. Also in the pure-inductive case, the matrix becomes symmetric by simply choosing $F(q) = -L\delta(q)$. Therefore all the eigenvalues of M are real, then the system is stable with the pure-capacitive and the pure-inductive wakes.

On the other hand, the pure-resistive wake brings a quite different situation. In this case we can use $F(q) = R(\theta(q)-1/2)$ which makes the second term of $M_{jmj'm'}$ in Eq. (18) antisymmetric by exchanging (j,m) and (j',m'). Thus the matrix M becomes antisymmetric except the diagonal elements. The main characteristics of the pure resistive case can be understood by looking at a 2 by 2 sub-matrix of the big matrix M. Let us pick up 2 by 2 elements of M which belong to the same azimuthal mode m and have different amplitudes j_1 and j_2. Such a sub-matrix has the form of

$$M_{j_1j_2} = m^2 \begin{pmatrix} \omega_{j_1}^2 & b_m(k) \\ -b_m(k) & \omega_{j_2}^2 \end{pmatrix} \tag{19}$$

where $b_m(k)$ is a quantity given by the integral in (18). The matrix $M_{j_1j_2}$ is unstable when

$$(\omega_{j_1}^2 - \omega_{j_2}^2)^2 - 4b_m(k)^2 < 0 \ . \tag{20}$$

If the frequencies of two amplitudes are equal or close to each other, *i.e.*, $\omega_{j_1} \approx \omega_{j_2}$, the sub-matrix $M_{j_1j_2}$ is unstable for any azimuthal mode number m. Then the entire matrix M can be unstable, if the contributions of other components more or less cancel to each other.

FIG. 1

In the case of the pure-resistive wake, the behavior of $\omega(J)$ is shown in Fig. 1(a). The function actually gives the same frequency for two different amplitudes. This situation suggests that the stationary solution with the pure-resistive wake is always unstable, and we confirmed it by the numerical calculation for the large

matrix M. Figure 2 shows the growth rate of several unstable modes for the pure-resistive wake obtained by the large-matrix method with Gaussian distribution (in this paper we used 60 mesh points in the range $0 \leq J \leq 8$, and azimuthal modes of $m \leq 5$). According to the analysis above, this instability can be understood as a coupling of coherent modes with the same azimuthal mode number m.

FIG. 2

It is remarkable that all the intensity dependences of the matrix M start from the order of k^2 in the case of the pure-resistive wake. As the result the growth rate of the pure-resistive instability also starts at the order of k^2, which agrees with Fig. 2. Then the growth of the instability can be very weak at low intensity. In particular in an electron ring, the actual threshold of the instability is determined by the balance between the growth rate and the radiation damping rate of the mode. Although this instability appears in any azimuthal mode number m, their growth and damping rates are different. We see in Fig. 2 that the growth rate is highest for $m = 2$ especially in the region $kR \gtrsim 5$. The threshold becomes lowest for the $m = 2$ mode with an estimation of the radiation damping rates for these unstable modes.

Next we discuss on the mixed case of the pure-inductive and the pure-resistive wakes, *i.e.*, $W(q) = R\delta(q) - L\delta'(q)$. In the analysis for the pure-resistive wake, we have made a hypothesis that the degeneration of the synchrotron frequencies for two different amplitudes makes the instability. If the hypothesis is true, the additional inductive part can suppress the instability by boosting the frequency spread in the bunch. Figure 1(b) shows the $\omega(J)$ for the pure-inductive wake. The frequency $\omega(J)$ for the pure-inductive wake starts below 1 at $J = 0$, and simply raises toward 1 as the amplitude J increases. Thus when we add the inductive term to the resistive term, the minimal point of $\omega(J)$ shifts left (toward smaller J). As we increase the inductive part more, the minimal point eventually vanishes, so that the frequency is always different for any two different amplitudes. The condition that $\omega(J)$ becomes simply increasing in J is equivalent to

$$\left.\frac{d\omega(J)}{dJ}\right|_{J=0} \geq 0 \,, \tag{21}$$

in the mixed case of the pure-resistive and the pure-inductive wakes. Now we have reached a hypothetical condition (21) which gives the stability criterion of the mixed case. To verify that Eq. (21) gives the threshold of the instability, we have to express it in terms of the intensity and the magnitudes of the wakes. So far we use a Gaussian bunch, but the method can be applicable to any distribution with minor changes. First we rewrite the Hamiltonian around its fixed point q_0 for the given intensity and wakes, using a new coordinate $\overline{q} \equiv q - q_0$:

$$\begin{aligned} H(p,\overline{q}) &= \frac{p^2}{2} + \frac{(\overline{q}+q_0)^2}{2} - \int_0^{\overline{q}} V_0(\overline{q}' + q_0) dq' \\ &= \frac{p^2}{2} + \frac{(\overline{q}+q_0)^2}{2} - kR \int_0^{\overline{q}} \rho_0(\overline{q}' + q_0) d\overline{q}' + kL\left(\rho_0(\overline{q}+q_0) - \rho_0(q_0)\right) \,, \end{aligned} \tag{22}$$

where we have used (7) and the mixed wake. From the definition of the fixed point q_0,

$$0 = \left.\frac{\partial H}{\partial \overline{q}}\right|_{\overline{q}=0} = q_0 - kR\rho_0(q_0) \,, \qquad \rho_0'(q_0) = 0 \,, \tag{23}$$

where Eq. (4) has been applied. We also assume Gaussian distribution

$$\rho_0(\overline{q}) = A \int_{-\infty}^{+\infty} \exp\left(-H(p,\overline{q})\right) dp \,, \tag{24}$$

where A is the normalization factor. To obtain the derivative (21), we need terms up to the fourth order of $\overline{q}$ in $H(p,\overline{q})$. Such an expansion of $H(p,\overline{q})$ can be obtained repeatedly by combining Eqs. (22), (23), and (24). The result is

$$H(p,\overline{q}) = \frac{p^2}{2} + \frac{a}{2}\overline{q}^2 + \frac{a^{5/2} q_0}{6}\overline{q}^3 + \frac{a^4 q_0 (q_0 + 3L/R)}{24}\overline{q}^4 + O(\overline{q}^5) \,, \tag{25}$$

where $a \equiv (R/(R + Lq_0))^{1/2}$. From (25) it is not difficult to derive the derivative

$$\left.\frac{d\omega(J)}{dJ}\right|_{J=0} = \frac{a^3 q_0}{24R}(9L - 2Rq_0) \,. \tag{26}$$

Thus the stability condition (21) is written in a simple form

$$kL \geq \frac{2}{9} kRq_0 \,. \tag{27}$$

Note that the equilibrium position q_0 is a function of kR and kL. Its lowest-order term is given by Eq. (23) as

$$q_0 = \frac{kR}{\sqrt{2\pi}} + O(k^2) \,. \tag{28}$$

Fig. 3

To examine the validity of the stability condition (27), we show the growth rate of the mixed wake, obtained by the large-matrix method, as a function of both kR and kL in Fig. 3. Here we draw the $m = 2$ mode which gives the highest growth rate among all unstable modes in most cases. We also show the curve given by Eq. (27) on top of that. The fixed point q_0 is obtained by solving Eq. (23) numerically. This figure shows that Eq. (27) which is derived from Eq. (21) gives a fairly well criterion of the stability. This result justifies our basic hypothesis that the degeneration of the synchrotron frequency for two different amplitudes is the source of the weak longitudinal single-bunch instability. Figure 3 and Eq. (27) also suggest that reducing the wakefield by smoothing the beam duct does not always improve the threshold of the instability, unless the resistive part is significantly reduced. Although the bunch-lengthening below threshold is improved, but the threshold itself is lowered by reducing only the inductive part of the wake. Even with a more general form of the wake potential, reducing the frequency spread inside a bunch can be dangerous in the point of view of the stability condition. In this point of view, an additional frequency spread with a higher harmonic rf accelerating voltage has a possibility to remove the instability.

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FIGURE CAPTIONS

1) The normalized synchrotron frequency $\omega(J)$ of the single-particle motion in the bunch as the function of the amplitude J. (a): the pure-resistive wake $W(q) = R\delta(q)$. (b): the pure-inductive wake $W(q) = -L\delta'(q)$.

2) Growth rates of unstable modes with the pure-resistive wake $W(q) = R\delta(q)$ obtained from the matrix (18). The parameter m specifies the nearest integer of the frequency of each mode. It is seen that the growth rate is roughly proportional to k^2.

3) Contour plot of the growth rate $\mathrm{Im}(\mu)$ of $m = 2$ mode of the mixed wake $W(q) = R\delta(q) - L\delta'(q)$, obtained from the matrix (18). The pitch of the contour is $\Delta\mathrm{Im}(\mu) = 0.006$. The dashed curve is the stability condition Eq. (27).

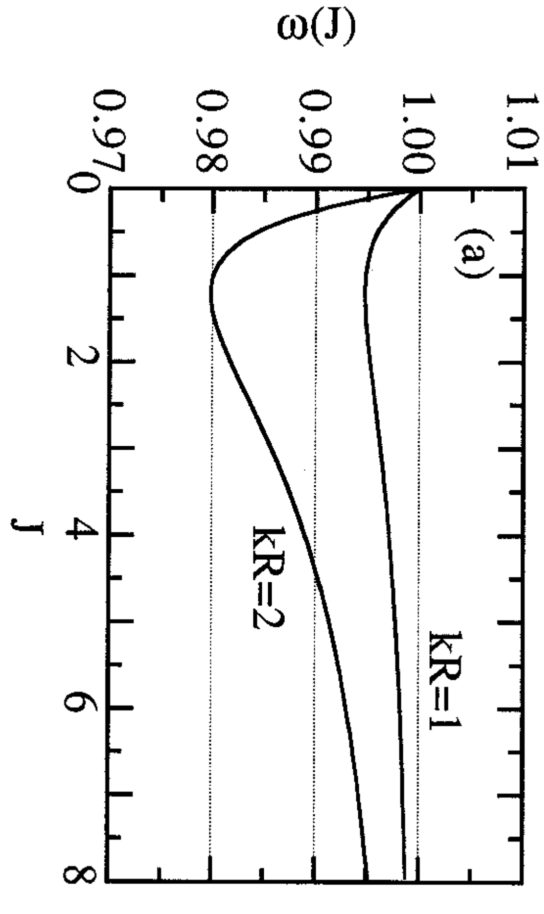
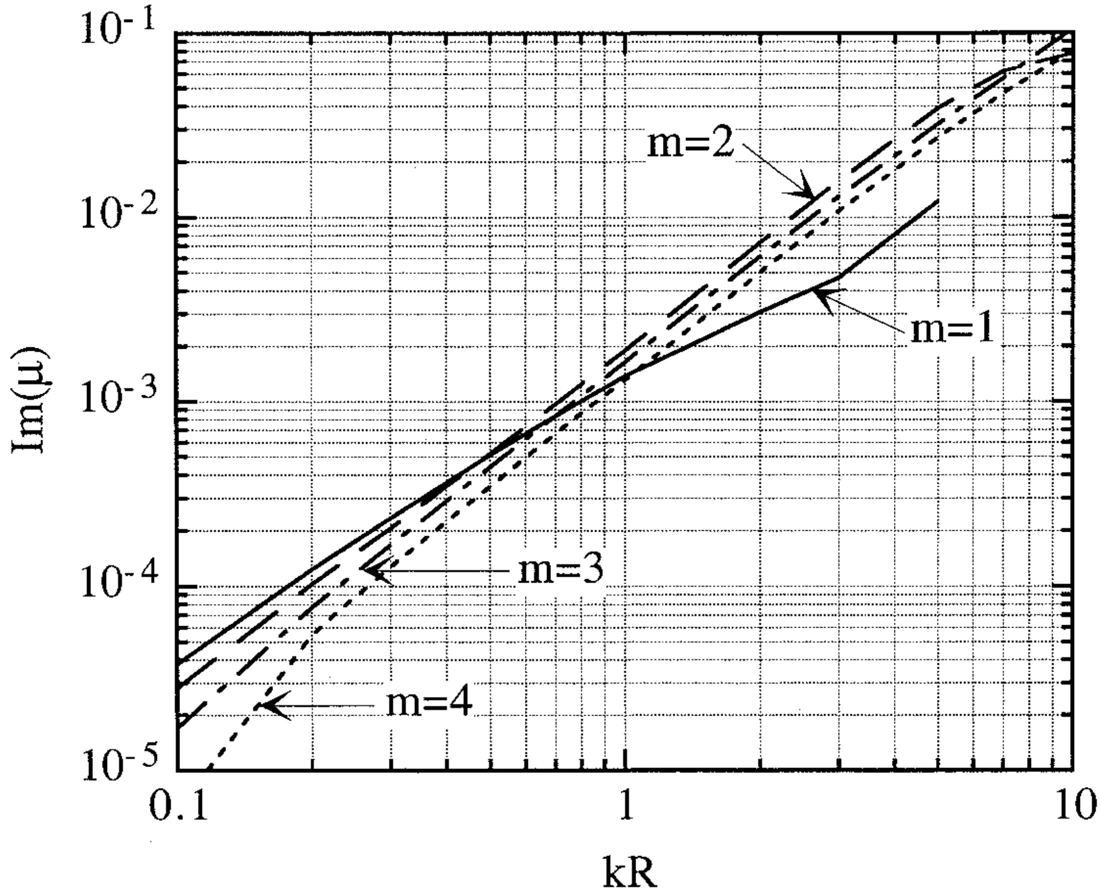
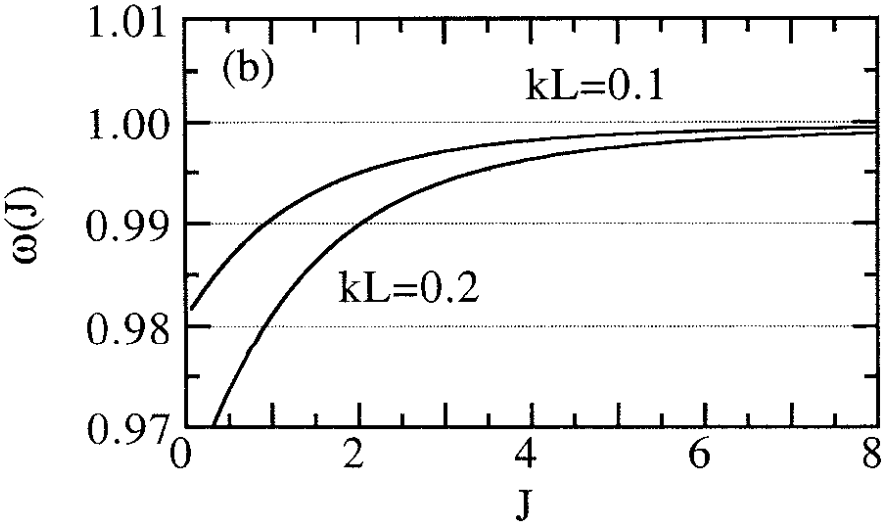


Fig. 1(a)



Im(μ)
10-1
10-2
10-3
10-4
10-5
0.1
1
10
kR
m=1
m=2
m=3
m=4

Fig. 2



ω(J)
1.01
1.00
0.99
0.98
0.97
0
2
4
6
8
J
(b)
kL=0.1
kL=0.2

Fig. 1(b)

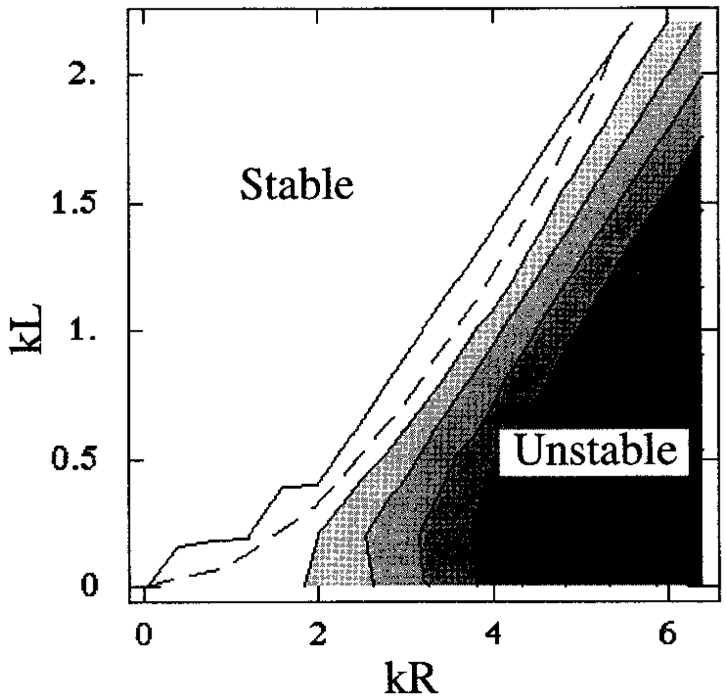


Fig. 3