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TUNE SHIFT OF COHERENT BEAM-BEAM OSCILLATIONS

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The coherent beam-beam oscillation is a good probe to the beam-beam interaction in storage rings. It can artificially be excited by a frequency-changeable kicker. The amplitude of the coherent oscillation can be very small, as long as it is above the sensitivity level of the detector, so that it does not disturb the beam-beam phenomena. The beam emittance can then be estimated by the observed tune shift of the coherent oscillation.

However, in order to make use of this, we need a relation between the tune shift of the coherent π-mode oscillation, $\Delta\nu_\pi$, and the so-called beam-beam parameter

$$\xi_{x(y)} = \frac{\beta^*_{x(y)}}{4\pi}\frac{2Nr_e}{\gamma\sigma^*_{x(y)}(\sigma^*_x + \sigma^*_y)}, \tag{1}$$

where $\beta_{x(y)}$ is the horizontal (vertical) beta function, r_e the classical electron radius, N the number of particles in a bunch, γ the beam energy in units of rest mass, $\sigma_{x(y)}$ the r. m. s. beam size and the asterisk denotes the values at the collision point. When there is one bunch per beam and there is only one collision point in the ring, the π-mode tune shift has been given by

$$\Delta\nu_\pi = \lambda\xi \tag{2}$$

with λ=2 and 1 by Piwinski[2] and by Hirata[3], respectively. (In this paper we discuss only the linear term in ξ.)

An intensive analysis has been done in the TRISTAN Accumulation Ring at KEK using Hirata's formula, which seems to be the better among the two. However, the value of the horizontal emittance estimated in this way has always been smaller by some 30 percent than the value computed from the design optics.

The purpose of the present paper is to solve this discrepancy by reconsidering the theory of the coherent beam-beam interaction. A similar analysis has been done by Meller and Siemann[4] for the vertical oscillation of very flat beams.

Horizontal Oscillation of Very Flat Beams

First let us discuss the horizontal coherent oscillation of very flat beams. It is a very special case because it is reduced to a one-dimensional problem.

We assume that there is one bunch per beam and one interaction point and that the two beams are equal in energy, population and optics.

The equation of motion of a particle in the positron bunch is $dx/d\theta = \nu_{0x}p_x$ and

$$\frac{dp_x}{d\theta} = -\nu_{0x}x - \delta_p(\theta)\frac{2Nr_e\beta^*_x}{\gamma}\int\frac{(x-x')\rho^{(-)}(x',y')}{[\sigma^*_x(x-x')]^2 + [\sigma^*_y(y-y')]^2}dx'dy', \tag{3}$$

where x and p_x are the normalised variables defined by

$$\theta \equiv \frac{1}{\nu_{0x}}\int_0^s \frac{ds}{\beta_x(s)}, \qquad x \equiv \frac{\hat{x}}{\sqrt{\epsilon_x\beta_x(s)}}, \qquad p_x \equiv \nu_{0x}\frac{\beta_x(s)d\hat{x}/ds + \alpha_x(s)\hat{x}}{\sqrt{\epsilon_x\beta_x(s)}}.$$

Here, s is the length along the design orbit, $\hat{x}(\hat{y})$ the ordinary horizontal (vertical) coordinate, ν_{0x} the unperturbed tune, β_x, α_x the unperturbed Twiss parameters, ϵ_x the unperturbed emittance and $\delta_p(\theta)$ is the periodic delta function with period 2π. The equilibrium distribution in the absence of the beam-beam interaction satisfies $\langle x^2\rangle = \langle p_x^2\rangle = 1$ and $\langle xp_x\rangle = 0$. The electron distribution function $\rho^{(-)}$ is normalised as $\int \rho^{(-)}(x,y)dxdy = 1$. We average $\delta_p(\theta)$ and replace it with $1/2\pi$.

In the limit of very flat beams, i.e., $R = \sigma_y^*/\sigma_x^* \to 0$ we get

$$\frac{dp_x}{d\theta} = -\nu_{0x}x - 2\xi_x \text{p.v.} \int_{-\infty}^{\infty} \frac{\rho^{(-)}(x')}{x-x'}dx',$$

where p.v. denotes Cauchy's principal value. The Vlasov equation for the phase space distribution function of the j-th beam $\Psi^{(j)}(p_x,x;\theta)$ (j=+,−) is

$$\frac{\partial\Psi^{(+)}}{\partial\theta} + \nu_{0x}\frac{\partial\Psi^{(+)}}{\partial\phi_x} = 2\xi_x \text{p.v.} \int \frac{\rho^{(-)}(x';\theta)}{x-x'}dx' \cdot \frac{\partial\Psi^{(+)}}{\partial p_x}, \tag{4}$$

$$\rho^{(-)}(x;\theta) = \int_{-\infty}^{\infty} \Psi^{(-)}(x,p_x;\theta)dp_x.$$

Here, we have introduced the (unperturbed) action-angle variables (I_x,ϕ_x) defined by $x = \sqrt{2I_x}\sin\phi_x$ and $p_x = \sqrt{2I_x}\cos\phi_x$.

Since we are considering the coherent oscillation with an infinitesimal amplitude (the incoherent oscillation amplitude is finite), we linearize eq.(4). We split Ψ into the equilibrium and the oscillation part as $\Psi^{(j)}(p_x,x;\theta) = \Psi^{(0)} + \psi^{(j)}(p_x,x;\theta)$ $(j = +,-)$. The equilibrium distribution $\Psi^{(0)}$ is a function of I_x only up to the first order of ξ_x, and we assume in the following the Gaussian distribution $\Psi^{(0)}(I_x) = e^{-I_x}/2\pi$. The linearized Vlasov equation is

$$\frac{\partial\psi^{(+)}}{\partial\theta} + \nu_{0x}\frac{\partial\psi^{(+)}}{\partial\phi_x} = 2\xi_x\left[\frac{\partial\psi^{(+)}}{\partial p_x}\text{p.v.}\int\frac{\rho^{(0)}(x')}{x-x'}dx' + \frac{\partial\Psi^{(0)}}{\partial p_x}\text{p.v.}\int\frac{\psi^{(-)}(p_x',x';\theta)}{x-x'}dx'dp_x'\right] \tag{5}$$

with $\rho^{(0)}(x) = \int_{-\infty}^{\infty}\Psi^{(0)}(I_x)dp_x = e^{-x^2/2}/\sqrt{2\pi}$. The perturbed part $\psi^{(j)}$ can be expanded in terms of $e^{im\phi_x}$. Since there is no coupling between the modes of different m up to the first order in ξ_x, we retain the dipole term (m =1) only and write

$$\psi^{(j)}(I_x,\phi_x;\theta) = \Re e^{i(\phi_x-\nu\theta)}e^{-I_x/2}f^{(j)}(I_x).$$

Substituting this expression into eq.(5), multiplying $\exp(-i\phi_x)$ on both sides and averaging over ϕ_x, we obtain

$$\lambda f^{(+)}(I_x) = Q(I_x)f^{(+)}(I_x) - \int_0^{\infty} G(I_x,I_x')f^{(-)}(I_x')dI_x', \tag{6}$$

$$Q(I_x) = \frac{1-e^{-I_x}}{I_x}, \qquad G(I_x,I_x') = e^{-(I_x+I_x')/2}\sqrt{\frac{\min(I_x,I_x')}{\max(I_x,I_x')}}.$$

One can easily verify that the Σ-mode solution is exactly given by $f^{(+)}(I_x) = f^{(-)}(I_x) = \sqrt{2I_x}e^{-I_x/2}$ with the eigenvalue λ=0.

In the following we will consider the π-mode and put $f(I_x) = f^{(+)}(I_x) = -f^{(-)}(I_x)$. Then, eq.(6) becomes

$$\lambda f(I_x) = Q(I_x)f(I_x) + \int_0^{\infty} G(I_x,I_x')f(I_x')dI_x'. \tag{7}$$

In order to solve eq.(7) we expand $f(I_x)$ using Laguerre polynomials

$$f(I_x) = \sum_{m=0}^{\infty} f_m u_m(I_x) \qquad \text{with} \qquad u_m(I_x) = \sqrt{I_x} e^{-I_x/2} \frac{L_m^{(1)}(I_x)}{\sqrt{m+1}}$$

which satisfies the orthonormality condition $\int_0^\infty u_m(I_x) u_{m'}(I_x) dI_x = \delta_{mm'}$. The Laguerre polynomial $L^{(\alpha)}(x)$ is defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{x^k}{k!} \qquad (\alpha > -1).$$

Then the integral equation (7) becomes a matrix equation for f_m;

$$\lambda f_m = \sum_{m'=0}^{\infty} (Q_{mm'} + G_{mm'}) f_{m'} \tag{8}$$

where

$$Q_{mm'} = \int_0^\infty Q(I_x) u_m(I_x) u_{m'}(I_x) dI_x, \qquad G_{mm'} = \int_0^\infty dI_x dI'_x G(I_x, I'_x) u_m(I_x) u_{m'}(I'_x).$$

After some manipulation we get

$$Q_{mm'} = \frac{1}{\sqrt{(m+1)(m'+1)}} \left[\min(m+1, m'+1) - \sum_{k=0}^{m} \sum_{k'=0}^{m'} \frac{(k+k')!}{2^{k+k'+1} k! k'!} \right]$$

$$G_{mm'} = \frac{1}{\sqrt{(m+1)(m'+1)}} \frac{(m+m')!}{2^{m+m'+1} m! m'!}.$$

If we truncate the expansion at order M, eq.(8) becomes a matrix eigenvalue problem. The convergence of the largest eigenvalue w. r. t. M is very good and we get λ =1.330 or $\Delta\nu_\pi = 1.330\xi_x$ for the horizontal oscillation of very flat beams.

(In addition to this solution, there is a continuum in $0 < \lambda < 1$ which corresponds to the incoherent oscillation. See [1] for the detail.)

The projected distribution $\rho(x;\theta)$ is plotted by the dashed line in Fig. 1 for the shift of the center-of-mass $X \sin \nu\theta$=0.1 sigma. (The full line is the equilibrium Gaussian distribution.) One finds that only the central part oscillates with the tail sitting still.

General Aspect Ratio

Next, we extend our formalism to the case of general values of the aspect ratio $R = \sigma_y^*/\sigma_x^*$. We will consider the horizontal oscillation only but the vertical oscillation can easily be obtained by exchanging x and y and replacing R by $1/R$.

By decomposing the four-dimensional phase space distribution function as $\Psi^{(j)} = \Psi^{(0)}(I_x, I_y) + \psi^{(j)}(I_x, \phi_x, I_y, \phi_y; \theta)$ with the equilibrium distribution $\Psi^{(0)}(I_x, I_y) = e^{-I_x - I_y}/(2\pi)^2$ and by taking the dipole term $\psi^{(j)} = \Re e^{i(\phi_x - \nu\theta)} e^{-(I_x+I_y)/2} f(I_x, I_y)$, we get the linearized Vlasov equation

$$\lambda f^{(+)}(I_x, I_y) = Q(I_x, I_y) f^{(+)}(I_x, I_y) - \int G(I_x, I_y, I'_x, I'_y) f^{(-)}(I'_x, I'_y) dI'_x dI'_y \tag{9}$$

with $\lambda = (\nu - \nu_{0x})/\xi_x$ as before and

$$Q(I_x, I_y) = \frac{1+R}{4\pi^3} \int d\phi_x d\phi_y \frac{\sin\phi_x}{\sqrt{2I_x}} \frac{(\sqrt{2I_x}\sin\phi'_x - x') \exp(-(x'^2 + y'^2)/2)}{(\sqrt{2I_x}\sin\phi_x - x')^2 + R^2(\sqrt{2I_y}\sin\phi_y - y')^2} dx' dy'$$

$$G(I_x,I_y,\,I'_x,I'_y) = -\frac{2(1+R)}{(2\pi)^4}e^{-(I_x+I_y+I'_x+I'_y)/2}\int d\phi_x d\phi_y d\phi'_x d\phi'_y$$
$$\times\frac{\sqrt{2I_x}\cos^2\phi_x(\sqrt{2I_x}\sin\phi_x-\sqrt{2I'_x}\sin\phi'_x)\sin\phi'_x}{(\sqrt{2I_x}\sin\phi_x-\sqrt{2I'_x}\sin\phi'_x)^2+R^2(\sqrt{2I_y}\sin\phi_y-\sqrt{2I'_y}\sin\phi'_y)^2}.$$

The terms involving ξ_y vanish by the average over ϕ_y.

One can confirm that eq.(9) has an exact Σ-mode solution λ=0 with $f^{(+)} = f^{(-)} = \sqrt{2I_x}e^{-(I_x+I_y)/2}$.

The π-mode equation is given by putting $f \equiv f^{(+)} = -f^{(-)}$ as

$$\lambda f(I_x,I_y) = Q(I_x,I_y)f(I_x,I_y) + \int G(I_x,I_y,I'_x,I'_y)f(I'_x,I'_y)dI'_x dI'_y. \tag{10}$$

The Laguerre expansion now takes the form

$$f(I_x,I_y) = \sum_{m,n=0}^{\infty} f_{mn}u_m(I_x)v_n(I_y)$$

where $v_n(I_y) = e^{-I_y/2}L_n^{(0)}(I_y)$. The matrix equation is then

$$\lambda f_{mn} = \sum_{m'n'}(Q_{mn,m'n'} + G_{mn,m'n'})f_{m'n'} \tag{11}$$

where the explicit forms of $Q_{mn,m'n'}$ and $G_{mn,m'n'}$ are given in [1].

The eigenvalue is plotted in Fig. 2 by the solid lines. (The Laguerre expansion is taken up to m+n≤M=20. The eigenvalues are believed to be correct to three decimal places.) The dashed line is an empirical fit by a second order polynomial of $r = R/(1+R) = \sigma_y^*/(\sigma_x^*+\sigma_y^*)$ using the three data, namely $\lambda(R=0)$=1.3298, $\lambda(1)$=1.2144 and $\lambda(\infty)$=1.2385. The resulting formula is

$$\frac{\Delta\nu_{x,\pi}}{\xi_x} \equiv \Lambda(r) = 1.330 - 0.370r + 0.279r^2, \qquad \frac{\Delta\nu_{y,\pi}}{\xi_y} = \Lambda(1-r). \tag{12}$$

In practice this formula is accurate enough as seen in the figure.

The eigenvalue is in a narrow range $1.21<\Lambda<1.33$ and is rather insensitive to the aspect ratio. It has a minimum near R=2 (r=0.67).

A precise measurement of λ for horizontal and vertical tune shift of a flat beam has been done at TRIATAN accumulation ring[6]. The agreement with our theory was excellent.

Meller and Siemann[4] have presented an analysis of the vertical oscillation of flat beams by a different approach using 'averaging method' which is equivalent to extracting the dipole mode in our way. The integral equation they got seems to be the same as ours but they quote the π-mode eigenvalue λ=1.34 to be compared with our value 1.2385. The difference is not large and is probably due to their limited matrix dimension. They also computed the Σ-mode and obtained λ=0.097 which has to be exactly zero. Presumably, this is due to the matrix dimension or to the misidentification of modes.

E. Keil[5] has presented the result of tracking studies of the vertical oscillation. He found that the π-mode tune shift is considerably lower than that of Piwinski's formula. It agrees with our theory qualitatively.

More General Cases

Consider beams with different beam-beam parameters. We still assume the equal beam size, one bunch per beam and only one interaction point. Let us denote the beam-beam parameter on the j-th beam by $\xi_x^{(j)}$ (j=+,−). One finds the coupled integral equation

$$\Delta\nu_x f^{(+)} = \xi_x^{(+)}(Qf^{(+)} - G\circ f^{(-)}), \qquad \Delta\nu_x f^{(-)} = \xi_x^{(-)}(Qf^{(-)} - G\circ f^{(+)})$$

where $G\circ$ is the operator $(G\circ f)(I_x, I_y) = \int G(I_x, I_y, I'_x, I'_y) f(I'_x, I'_y) dI'_x dI'_y$ and Q and G are the same functions as before.

The Σ-mode solution is the same as before because $Qf - G\circ f = 0$ for that function. On the other hand, the π-mode cannot be separated and we have to solve numerically. We have employed the same Laguerre expansion (truncated at M=20, which gives the matrix dimension 462×462). The resulting eigenvalue is plotted in Fig.3a. Here, $\Delta\nu_x$ is normalized by $(\xi_x^{(+)} + \xi_x^{(-)})/2$ and the horizontal axis is $r = R/(1+R)$. (We omitted the curves for $\Delta\nu_y$ because they are merely a reflection w. r. t. the line r=1/2.) Each curve corresponds to a fixed value of $R_\xi \equiv \xi_x^{(-)}/\xi_x^{(+)}$. (Obviously, R_ξ and $1/R_\xi$ give the same eigenvalue.)

The same data is plotted in a different manner in Fig.3b where the horizontal axis is R_ξ and each curve represents different r. The curve for $R = \infty$ seems to agree qualitatively with that in ref[4]. One finds that the curves in Fig.3b are rather flat when R_ξ is not far from unity, which means

$$\Delta\nu_x \sim \frac{1}{2}(\xi_x^{(+)} + \xi_x^{(-)})\Lambda(r) \qquad (\frac{1}{2} \lesssim R_\xi \lesssim 2) \tag{13}$$

is a reasonable approximation.

It is easy to generalize our formalism to the case of N_b bunches per beam with $2N_b$ interaction points. Usually the rigid bunch model gives N_b modes, i.e., Σ-mode, π-mode and $N_b - 2$ intermediate modes. It turned out that all the intermediate modes are incoherent modes actually. They do not show sharp spectrum but merely give a continuum in $(0, 2N_b\xi)$. The Σ-mode is trivial and the only significant mode is the π-mode whose tune shift is given by $\Delta\nu = 2N_b\xi\Lambda(r)$ with the same function Λ.

References

[1] K. Yokoya, Y. Funakoshi, E. Kikutani and H. Koiso, *Tune Shift of Coherent Beam-Beam Oscillatins*, submitted to Part.Acc.

[2] A. Piwinski, *Observation of Beam-Beam Effects in PETRA*, IEEE Trans. NS-26, page 4268, 1979.

[3] K. Hirata, *Coherent Betatron Oscillation Modes due to Beam-Beam Interaction*, Nucl.Instr.Meth. **269** (1988) 7.

[4] R. E. Meller and R. H. Siemann *Coherent Normal Modes of Colliding Beams*. IEEE Trans. NS-28, page 2431, 1981.

[5] E. Keil, *Visible Frequencies in the Nonlinear Coherent Beam-Beam Effect*, LEP Note 294, Apr.1981.

[6] H. Koiso *et al. Measurement of the Coherent Beam-Beam Tune Shift in the TRISTAN Accumulation Ring*, these proceedings.

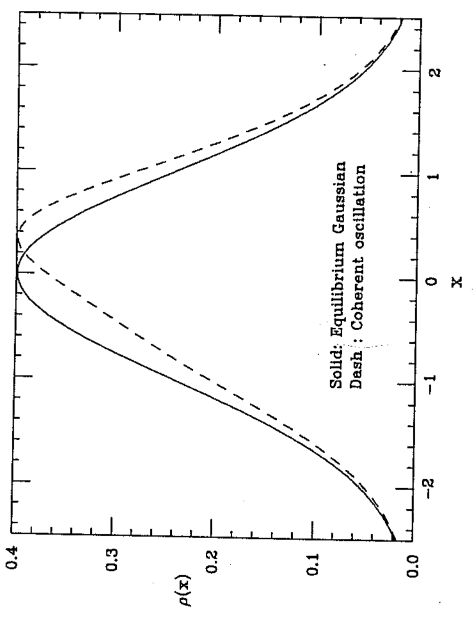

Fig.1 Projection on to X–axis. Amplitude=0.10

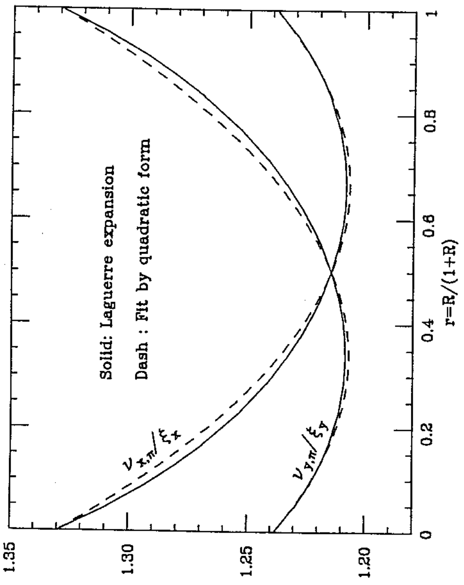

Fig.2 Eigenvalue to 2D Equation

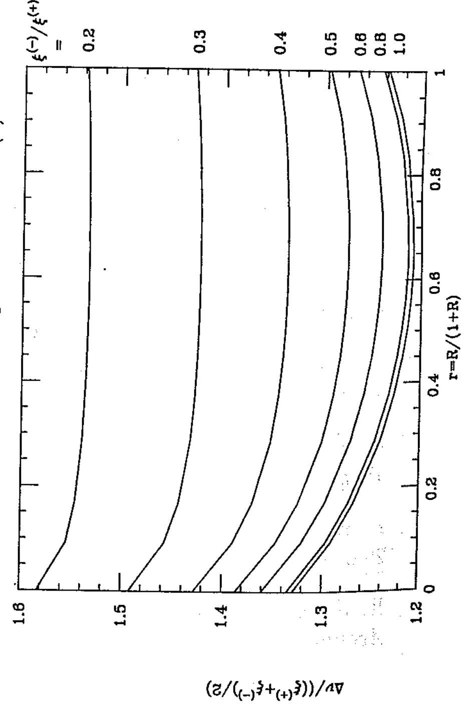

Fig.3a Eigenvalue for Unequal Beam Case (1)

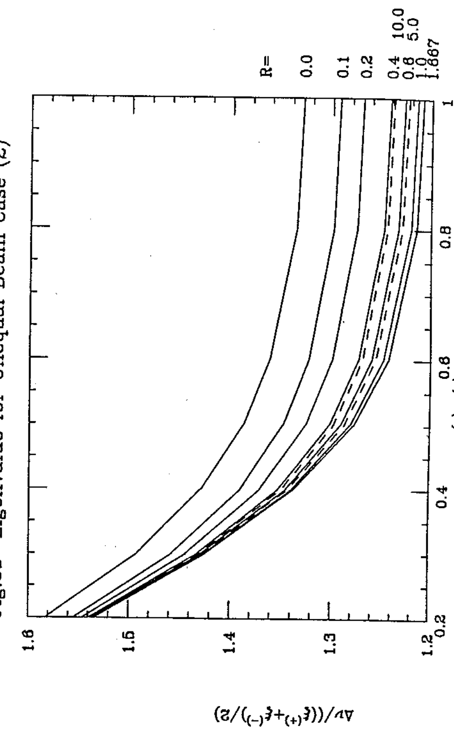

Fig.3b Eigenvalue for Unequal Beam Case (2)