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Configuration of reactive integrated wake forces

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The integrated wake-forces for an ultra-relativistic beam are usually constrained to a special canonical configuration in r and θ, assumed to be valid completely independent of the particular boundary. It will be shown that for (reactive) wakes in structures with non-constant cross-section this canonical configuration is too restrictive. Even in a round structure an m=0 mode or an on-axis beam can very well cause transverse momentum kicks for off-axis trailing particles. Also numerical integration of the accelerating voltage across a cavity should be done on the axis, the assumed scaling laws to other radial positions being invalid.

I. Introduction

In different texts on accelerator physics ([1] (Fig. 2.1 a-c, referenced p. 57), [2], [3] p. 78f, [4]) it is shown that in general, and independent of the particular boundary, when particles moving at the velocity of light travel through a beam pipe the m^{th} multipole wake always scales in the radial direction as r^m. In particular this means that the m=0 monopole wakes are considered to be constant, independent of r, and hence, due to the Panofsky-Wenzel (PW) theorem [5], they do not create any transverse momentum kick.

These studies – except for [3] – rely on the same common fundamental feature: it is always assumed that the integrated longitudinal force is transmitted uniquely by so-called synchronous waves. These synchronous waves, whose properties are exploited only implicitly in some calculations, are traveling waves of different frequencies but all having a phase velocity, v_ϕ, matching the particle speed $v_p=v_\phi \to c$. Contributions from any other traveling wave with different phase velocity – which may very well be excited by the leading particle – average away. In the only study not relying on synchronous waves, the method of the 'generalized integration contour' [3] is used. Here, a term with an assumed asymptotic behavior as $1/\gamma^2$ is neglected; in reality it has a constant asymptotic behavior. These studies will be analyzed later in more detail showing that they are all invalid for structures with non-constant tube cross-sections.

It is well known that in structures *with changes in cross-section*, there exist fields other than traveling waves, the so-called evanescent or attenuating modes (or space-harmonics), which cannot be 'decomposed' into traveling waves as is the case for 'classical' standing waves. These fields have a different radial behavior for the m^{th} multipole than a simple r^m. These "evanescent" fields can be superimposed onto the classical traveling waves to create fields that are *on average* synchronous – i.e. very well able to produce non-vanishing forces – but which are different from true synchronous waves and which have a very different radial configuration.

Therefore in this paper the radial behavior of wake fields in perfectly conducting but non-smooth structures will be examined and it will be shown that the canonical configuration is not guaranteed in general.

A lack of proof does not mean that a theorem is invalid: another proof may be constructed. To settle the case once and for all, an example of an infinitely long circular structure with a perfectly cylindrical symmetric (m=0) field will be analysed to show that the integrated longitudinal force clearly depends on the radial position of the (trailing or test) particle, in definite contradiction to common convictions.

II. Present convictions

The integrated wake forces for v_p=c particles in a structure with a metallic boundary are generally presented (e.g. [1]) in a canonical configuration as

$$\int_{-\infty}^{+\infty} ds \cdot \vec{F}_{\|}^{(m)} = -eI_m W'_m(z) \cdot r^m \cdot cos(m\Theta) \quad (1a)$$

$$\int_{-\infty}^{+\infty} ds\, \vec{F}_{\perp}^{(m)} = -eI_m W_m(z) \cdot m \cdot r^{m-1} \cdot (\ddot{\mathbf{r}} \cdot cos(m\Theta) - \ddot{\Theta} \cdot sin(m\Theta)) \quad (1b)$$

This is very surprising at first glance since any multipole m has a single contribution proportional to r^m exclusively and no further contributions with other powers of r exist, as one could naively expect. On top of this, this behavior is not only valid for a special boundary but is completely independent of it. Further calculations, based on this canonical configuration, are considerably simplified and the results are readily used in different contexts. For example, it is claimed that an on-axis beam in a round structure does not produce transverse momentum kicks even for off-axis trailing particles.

The *special case m=0* of (1) is simply

$$\int_{-L/2}^{L/2} ds \cdot \vec{F}_{\|}^{(0)} = -eI_o \cdot W_0'(z) \quad (2a)$$

$$\int_{-L/2}^{L/2} ds \;\; \vec{F}_{\perp}^{(0)} \;\; = 0 \quad (2b)$$

In any beam pipe with a *circular* boundary (but where the circular cross-section may change in the longitudinal direction) any contributions from modes m>0 are excluded for an exciting *beam on axis*. This means that the integrated longitudinal force seen by the trailing particle, (2a), is completely independent of its radial position, and the integrated transverse force, (2b), the transverse momentum kick, is identical to zero (in agreement from (2a) with PW).

On the other hand, the fields believed to be at the origin of the wakes are solutions of Maxwell's equations in cylindrical coordinates having an E_z-component of the type $J_m(k_r \cdot r) \cdot \cos(m\theta) \cdot \cos(\omega \cdot t - k_z \cdot z)$ (see Appendix A1). The monopole function $J_0(x)$ can be represented as a globally convergent infinite series of powers of x, the lowest two terms being $1 - x^2/4$. Hence, for $J_0(k_r \cdot r)$ the lowest terms are $1 - (k_r/2)^2 \cdot r^2$. This justifiably raises the question as to why in (2a) an r^2 term does not exist, and this *in general* and for *any* (round) boundary.

The fundamental line of thought evoked (partly implicitly) is straightforward: The leading particle may very well excite many waves of the form $E_z = A \cdot J_0(k_r \cdot r) \cdot \cos(\omega \cdot t - k_z \cdot z)$ with many different k_z. The trailing particle is assumed to travel with $v_p = c$, hence the longitudinal integrated wake force $F_{\|}$ due to such a partial wave of amplitude A along a path from $z = L_1$ to $z = L_2$ is

$$F_{\|} = e/c \cdot A \cdot J_0(k_r r) \int_{L_1}^{L_2} dz \;\; \cos((\omega/c - k_z) \cdot z) \quad (3)$$

When increasing the length of the integration path, $F_{\|}$ will oscillate with constant amplitude but remain bound for all k_z except $k_z = \omega/c$. In the latter case the integrand is cos(0)=1 and $F_{\|}$ will increase linearly with path-length, dwarfing more and more all other contributions (delta-function). Only this special wave has to be considered as contributing to the wake forces.

The parameters k_r and k_z are not independent. To make such functions valid solutions of Maxwell's equations, the *frequency constraint* (Appendix A1)

$$(\omega/c)^2 \;=\; k_r^2 \;+\; k_z^2 \quad (4)$$

has to be respected. In the above special case $k_z = \omega/c$ it follows that $k_r = 0$ and hence all powers of $k_r \cdot r$ disappear, with the exception of the constant term. This is then considered as proof of the configuration (2); in essence, "contributions from all non-synchronous waves average away". It is easy to show that this line of thought leads to the same reasoning for any m>0 where the synchronous wave has a single non-vanishing coefficient for r^m, and is considered as proof for (1).

II.1 Review of the proof

This proof assumes implicitly that all Maxwellian field patterns can be expressed by the superposition of infinite *traveling* waves, i.e. waves that travel without any change of phase and amplitude from z=-∞ to z=+∞, and that (for any frequency) there exists one special such traveling wave, the synchronous wave, that is exclusively capable of transmitting non-zero integrated longitudinal forces onto particles with $v_p = c$. Therefore this latter wave imposes its radial behavior onto all non-vanishing integrated wakes, longitudinally and, via PW, transversely.

However, it will be shown in the following that there also exists an infinity of Maxwellian fields that transmit non-zero integrated longitudinal forces but do not have the canonical radial configuration (1). These fields can be viewed as a superposition of multiply-reflected *elementary plane* waves (as are light rays); they are *not (infinite) traveling* waves as exclusively assumed by (1).

In the following chapters the above claims will be elaborated. This will be done using elementary physics means first, including a definite counter-example to (1); this eliminates any hope of saving (1). Then a more elaborate analysis will be done on the 'additional' fields. It will be demonstrated how RF acceleration can be achieved even without synchronous waves by having "on average" synchronous fields.

Then the 'proofs' given in the literature will be analyzed to show that they are not valid for structures with non-constant cross-section.

Finally the implications of these necessary changes of concept will be discussed.

III. Elementary physics methods

All Maxwellian fields can be expressed as the superposition of plane waves [6]. Such plane waves, as for light rays, traveling in the z-direction only have E and B field components perpendicular to the direction of propagation, and hence, while they can be synchronous with a $v_p = c$ particle, they have no local nor integrated longitudinal interaction.

In free space these waves travel unhindered but when (perfectly[a]) reflecting surfaces are present they may be reflected to and fro between these surfaces, hence interrupting the otherwise infinitely long path.

A simple case [6] is, for example, an infinite

[a] all surfaces are implicitly assumed perfectly conducting in this paper

metallic plane hit by a plane wave with an angle of incidence α. Luckily all reflections add up to something simple, a new *plane* wave traveling in the 'mirrored' direction π-α. Best of all, incident and reflected plane waves can be combined mathematically to something that appears again like a *single* wave traveling from -∞ to +∞ with a propagation speed different from c. Since the elementary waves travel in an inclined direction, the energy propagation velocity of the combination, the group velocity, is $u=c\cdot\cos(\alpha)$, hence less than c. The 'phase velocity' v with which maxima and zeros appear to travel along the metallic plane [6], is $v=c/\cos(\alpha)$, and hence is no longer synchronous.

The combination has a new feature: due to the projection of the field components onto the now inclined common axis (angle $\pm\alpha$) longitudinal field components appear.

This interesting feature might be summarized as: either the wave is synchronous but has no longitudinal interaction or it has a local longitudinal interaction but loses synchronicity; in any case the integral is zero.

When the reflecting surface becomes a smooth tube – i.e. it has the same cross-section for any z – the to and fro reflections of the elementary plane waves can luckily again be grouped mathematically into traveling waves from –∞ to +∞ that can be expressed by the well-known functions, $\cos(m\cdot\theta)$ in the azimuthal direction, $\cos(k_z\cdot z)$ in the axial direction, $J_m(k_r\cdot r)$ (for E_z) in the radial direction and $\cos(\omega\cdot t)$ in time. Since $\cos(k_z\cdot z)\cdot\cos(\omega\cdot t)$ can be written as a linear combination of $\cos(\omega\cdot t\pm k_z\cdot z)$, *all* Maxwellian fields in a *smooth* tube can be expressed by linear combinations of these traveling waves.

The question can be asked: when all these functions describe Maxwellian fields in simple *smooth* tubes, how does one describe fields when someone makes a dent in any of those tubes? It raises the suspicion that there must be many more Maxwellian solutions than the above traveling waves to cope with the infinity of possible dents in the surface of any of those tubes.

In fact for each single traveling wave solution there exists an infinite set of further solutions of Maxwell's equations that are, mathematically speaking, linearly independent, hence can *not* be decomposed into traveling waves. Field patterns have to be expressed as a superposition of *all* these solutions.

These required, additional, solutions appear when 'opening a new dimension in the parameter space' by using complex k, but always respecting (4). Derivations can be found in Appendix A1. The full two-dimensional complex k_z-parameter space is shown in Fig. 1.

All solutions with purely real k_z behave as traveling waves, i.e. are proportional to $\cos(\omega t-k_z z)$, but for $|k_z|\geq\omega/c$ there is no r for which E_z is zero (see appendix A1), hence these solutions cannot physically exist in a (perfectly conducting) smooth tube. The 'classical' traveling waves with $|k_z|<\omega/c$, physical solutions in a smooth tube, are present in Fig. 1 as a one-dimensional line of limited length. The endpoints of this line present the two synchronous waves at $v_p=\pm c$.

The 'additional' non-traveling solutions with $\mathrm{Im}(k_z)\neq 0$ are the 'evanescent modes[b]', 'attenuating modes[c]' or even "space-harmonics". They are superimposed onto the traveling waves to form 'fringe fields' as soon as e.g. a wave guide has a bend or twist, or an obstacle such as a tuning post or an iris exists.

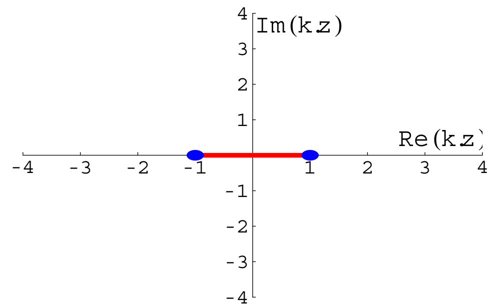

FIG. 1: The two-dimensional k_z parameter space in the complex plane (units ω/c). Requiring $\mathrm{Im}(k_z)=0$ and $|k_z|\leq\omega/c$ (red line), 'classical' traveling waves are only a tiny sub-set; the endpoints (blue dots) are the positive and negative synchronous waves. For *any* single traveling wave solution there exists an *uncountable* number of other solutions, nearly all having a non-zero integrated interaction with $v_p=c$ particles.

But the complexity of all possible fields in all arbitrary structures is so large that it is (in the general case) not possible to write down a *globally valid unique expression* using 'common' functions such as cos, exp or Bessel functions. In [7] it says: "… the exact solution of Maxwell's equations for the field distribution existing near a discontinuity is very difficult, if not impossible to obtain". In this case field matching [7][8] is used to find a description of the field everywhere in the structure. The whole structure is split into a, frequently infinite, series of adjacent limited volumes. Each volume has common (matching) surface(s) with its neighbor(s). In each volume the field is expressed as an 'infinite' series of 'common' functions, i.e. using the *full* set of possible complex k_z, locally respecting Maxwell's equations and offering *all* possible solutions in any volume. Matching ensures that Maxwell's equations are guaranteed across the matching surfaces[d].

[b] as in [7], p339
[c] as in [11]
[d] In a loose way this procedure can be compared to the Taylor development of an (analytic) function over -∞ to +∞.

The sum of all these local solutions then describes the field in the whole structure. Field matching is not, as one might think, a 'numerical fudge' to get an approximation solution, but is the only way to have the *exact* (read: reasonably precise) field description using everywhere a limited number of 'common' functions.

Now one of the initial riddles, which has plagued the author for a long time and partly triggered this work, can be solved. In a single cavity with two infinitely long beam tubes the tubes definitely carry no field for the trapped modes of the cavity, i.e. modes below cut-off frequency. How then can a hypothetical synchronous wave, claimed to be the unique carrier of the integrated force, be present in the empty tube, especially since a solution with $k_z=\omega/c$ cannot exist in a smooth, perfectly conducting, tube? Taking the claim of the unique synchronous wave literally, RF acceleration should be impossible.

However, by including the 'additional' solutions with $Im(k_z)\neq 0$, all pieces fall into place. There is a *segment* of the classical standing waves (i.e. opposing traveling wave pairs) in the main cavity volume. Fringe-fields, composed of evanescent modes (the 'additional' solutions), essentially localized around the cavity-tube transition, form a field pattern that – together with the standing waves – complies with Maxwell's equations and connects the field volume of the cavity with the empty tube volume(s). The integrated force is 'created' only over a short range so that 'synchronous' loses its primary sense and it would need very special circumstances such that an integral like (3), with bounds that are in practice finite, comes up with a zero-value. This then finally agrees with the fundamental beam-loading theorem [9], "allowing" RF acceleration.

In the above single cavity example one might now execute a double Fourier transform on the total field pattern. The first transformation is spectral, separating each ω-component of the field; the second separates the azimuthal modes m (which includes the full modal spectrum of this cavity). Then, for each such mode, a path from $z=-\infty$ to $z=+\infty$ with any chosen r can be Fourier analyzed in the z-direction yielding Fourier coefficients $f_{\|}(\square_z,r)$, each representing a 'wave' of wavelength $\lambda_z=2\pi/k_z$ along the z-direction. It would appear that the traveling waves are re-appearing Phoenix-like out of the ashes, but this is not true: these Fourier coefficients are simple mathematical objects without direct physical meaning.

Since an infinite number of solutions with non-zero integrated force make up the coefficients $f_{\|}(\square_z,r)$, one cannot, without justification, relate $f_{\|}(\square_z,r_1)$ and $f_{\|}(\square_z,r_2)$ by simply imposing a 'scaling law' which would be found for the synchronous wave $f_{\|}(\square_{sync}=\omega/c,r)$. Detailed knowledge of the field map with all the modes involved – and with it the specific boundary – has to be incorporated.

Claiming that all solutions can be expressed by traveling waves alone is equivalent to expressing an N-dimensional vector by the basis vectors of an n-dimensional sub-space with n<<N. This works only under very special conditions – here for smooth tubes – but not in general.

IV. Refutation by a counter example

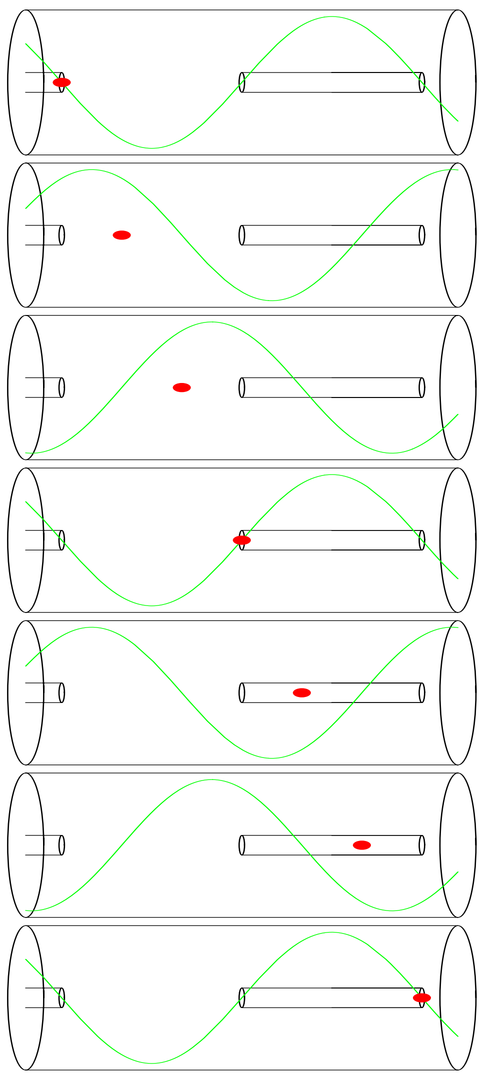

FIG. 2(a) – 2(g): Drift tube structure with $v_p=c$ and phase velocity $v=2\cdot c$ during one oscillation. The traveling RF wave is presented as a green sine-like line, the particle as the red dot. Time is running from top to bottom.

It can be done around any chosen point x_0, with an infinite number of terms perfectly correct for all x. But many terms (read: cannot be done easily with common functions) are necessary for a good precision in a region far away from x_0. Then it is better to use a different development around x_n (read: use the n^{th} local field-map).

The facts described above invalidate the given 'proof' for (1). The burden of proof is now again with the proponents of (1), either to deliver a different proof or to abandon the claim.

To settle the case once and for all – to disprove (1) – a counter-example will be shown. This counter example 'happens' in a perfectly round drift tube structure (Fig. 2) and a short explanation of its working principle follows.

IV.1 The drift tube structure

A drift tube structure [10] consists of a large tube of 'infinite' length, into which a non-synchronous m=0 wave is injected. Concentric with the beam axis are regularly spaced shorter sections of much narrower, hollow, drift-tubes interrupted by gaps, through which the beam passes. The length of tubes and gaps is adjusted such that a $v_p=c$ particle passes through a tube, hiding it from the wave when it would cause deceleration, and then passes by a gap when the wave can provide acceleration. This is sketched in Fig. 2(a) – 2(g).

At the tube-gap transitions there have to be 'fringe fields' composed of 'evanescent modes', to fulfill Maxwell's equations.

This type of a structure is used in the main (traveling wave) RF system of the CERN-SPS at 200 MHz and supplies up to 8 MV per turn.

IV.2 The counter-example

A perfectly round drift-tube structure, as in Fig. 2, is assumed[e] with cylindrically symmetric m=0 fields excited exclusively. The special case (2) is then applicable and the integrated longitudinal force should be independent of the radial position where the particle passes.

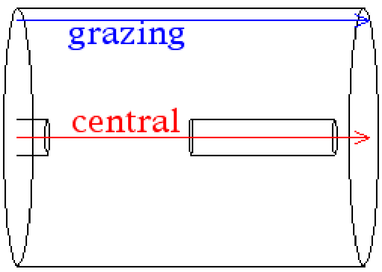

FIG. 3: Different paths along the drift tube structure, central $V_{acc}>>0$ and grazing $V_{acc}\rightarrow 0$.

A drift-tube structure is designed for a beam passing only inside the drift tubes. However, since there are no obstacles between drift tubes and main tube, there exist other possible particle paths completely inside the structure and never crossing any metallic boundary. The path inside the drift tubes shows efficient acceleration as demonstrated above. On the other hand a path grazing the large tube's inner surface, where the longitudinal E_z component has to approach zero, has asymptotically zero acceleration (see Fig. 3).

This radial behavior is in clear contradiction to (2) which implies that a perfectly cylindrically symmetric field should accelerate a $v_p=c$ particle with the same strength at any r. Hence the longitudinal integrated force (called acceleration in this context) is definitively not produced by the cited synchronous wave(s).

One claim might be made to save the situation: since there is a metallic interruption of the total volume at the drift tube level, one inner and one outer synchronous wave might exist with radially constant acceleration in each sub-volume (i.e. zero outside and V_{acc} inside the drift tubes) and with a discontinuity across the drift tube walls at radius r_d. However, one can easily show from first principles that the 'transition' across the drift tube is continuous (Fig. 4). One can assume the walls of the drift tubes infinitely thin so that the longitudinal integrated forces at $r_d-\delta r$ and at $r_d+\delta r$ exist, even for the smallest δr.

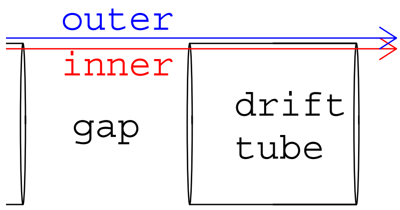

FIG. 4: paths just inside (r-δr) and just outside (r+δr) of the drift tube: continuous 'transition' for $\delta r\rightarrow 0$

Comparing two such particle paths, there are two different types of segments for the E_z-integration along the path. Due to the continuity of Maxwell's solutions, the integration along the gap segment of both paths will approach each other asymptotically as $\delta r\rightarrow 0$ for the *longitudinal* acceleration. Along the drift tubes the field becomes purely radial on both sides and E_z asymptotically disappears. This means that corresponding integral parts at $r_d-\delta r$ and at $r_d+\delta r$ asymptotically become identical for $\delta r\rightarrow 0$, the transition is *continuous*.

It is impossible then that acceleration is (only) supported by synchronous waves, there must be other field patterns – inside, outside or on both sides of the drift tubes – that are capable of producing non-zero integrated interactions and which, in contrast to (2), have radial dependence.

Equation (2) is a special case of (1), hence any counter-example to (2) also will contradict (1). This

[e] Lateral stems to mechanically hold the drift tubes have no electrical function and may be omitted RF wise [10]

shows without doubt that (1) is void.

V. RF acceleration

The synchronous wave, as used to create a configuration like (1), has an E_z-component that is radially proportional to r^m for the m^{th} azimuthal mode, i.e. is constant for m=0. In contrast to functions such as $J_0(k_r r)$, there is no zero at any r. Therefore such a wave cannot exist in any (perfectly conducting) smooth tube where $E_z=0$ is required on the boundary. This apparently prevents RF acceleration in any accelerator with at least a single piece of smooth beam tube (i.e. in all of them).

Without following up this odd observation it is in any case important to see how RF acceleration is realized without a synchronous wave. One example was already given with the drift-tube structure. An even more instructive example of *on average* synchronous fields will now be given. But to do so, the case of a single disk in an infinite, otherwise smooth tube, will be briefly shown first.

V.1 Tube with single iris

In Fig. 5 an (infinitely long) round tube is shown with a single iris, a disk with a central hole.

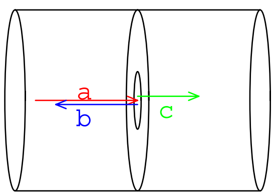

FIG. 5: Tube with single iris, containing a wave with incident amplitude a, reflected amplitude b and transmitted amplitude c. Energy conservation requires $a^2=b^2+c^2$.

It is known that an incident traveling wave[f] coming from -∞ along a smooth wave guide and hitting an 'obstacle', only partly gets past the obstacle (the transmitted wave). Another part gets reflected, traveling back towards the source. 'Far away' from the obstacle, only pure traveling waves can be detected.

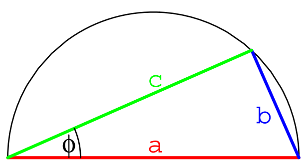

FIG. 6: Amplitude constraint for incident (a), transmitted (c) and reflected (b) waves, defining the phase-jump angle ϕ.

Due to energy conservation the sum of the squares of transmitted (c) and reflected (b) waves has to equal the square of the incident (a) wave amplitude, $a^2=b^2+c^2$. This can be drawn in the complex plane with a 'Thales circle' as in Fig. 6. The transmitted wave is therefore necessarily phase shifted by an angle ϕ with respect to the incident wave. The value of ϕ cannot be predicted easily, it depends on the details of the problem (see cited comment in [7])

This phase-jump produces a field discontinuity at the open part of the iris, a priori incompatible with Maxwell's equations. Again fringe fields are important.

These fringe fields do not travel – they have to 'sit tight' around the obstacle, oscillating with ω – and have to vanish (rapidly) away from the obstacle – they are not present 'far away'. This portrays just the type of 'additional' solutions that will be shown later. These solutions have only the apparent problem that they grow exponentially in at least one z-direction. However, this is not a problem in principle, they are only used locally in limited regions for field matching [7][8].

V.2 Disk loaded wave guide

Assuming a *non-synchronous* wave traveling as $\cos(\omega\cdot t - k_z\cdot z)$ with the phase velocity[g] $v=\omega/k_z>c$, one might imagine 'modulating' this wave along the z-axis proportionally to $\cos(\alpha\cdot z)$ with some α. The product $\cos(\omega\cdot t - k_z\cdot z)\cdot\cos(\alpha\cdot z)$ can be split into the sum of two terms $\cos(\omega\cdot t-(k_z\pm\alpha)\cdot z)$ – called the slow and fast wave in RF circles. By an adept choice of α one of them can be made to be synchronous with the particle. Of course such a hypothetical modulated wave does not comply with Maxwell's equations as such.

However, ways of modifying this basic idea to produce RF structures that accelerate $v_p=c$ particles perfectly well have been found by *using non-synchronous waves*. One method has already been shown: by approximating the $\cos(\alpha\cdot z)$-modulation with a step function, switching it on and off, in the drift tube structure. Another method consists of replacing the continuous cos-modulation by enforcing a repetitive phase-jump of this wave across an iris (see above), always by the same amount ϕ. Then in an infinite series of regularly spaced irises the field in the space between iris n and its next neighbor (forming cell n) has the global phase-factor $\exp(i\cdot n\cdot\psi)$. In this set-up the reflection of the back-reflected wave at the previous iris has also to be considered. ϕ and ψ are not equal since the phase-advance of the traveling wave

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[f] Above cut-off frequency; for simplicity the guide is assumed not to be 'overmoded', i.e. it carries only a *single propagating* mode of the chosen frequency.

[g] No contradiction to relativity, v being the *phase* velocity; the field energy propagation (*group*) velocity u has to be and always is ≤ c

from one iris to the next has also to be accounted for. Since n is (step-wise) proportional to z, this approximates on average a cos(α·z) modulation. The phase-jump at the iris produces a discontinuous field pattern, but, as previously shown, the superposition of fringe fields makes the total field comply with Maxwell's equations and the boundary conditions. As a result the electric field in the cell is a maximum when the particle passes the cell centre, i.e. the wave is *on average synchronous* without being 'the' synchronous wave. In [11] such a structure has been analytically calculated using field matching in a periodic structure, essentially needing only to solve one matching problem.

These accelerating structures, called 'disk-loaded wave-guides', are operated worldwide. One variant works in the well-known SLAC linac in the 2π/3 mode (the fields from cell to cell are phase-shifted by 2π/3) accelerating both v_p=c electrons and positrons there.

VI. Evanescent mode example

One special evanescent mode pattern is well known: this is for the field penetrating a tube (e.g. close to a cavity) while exponentially decreasing in field strength as its frequency is below the cut-off frequency of the tube. This field cannot be described by Maxwellian traveling (nor classical standing) waves; it is an independent type of solution as shown above.

As a concrete example, the full description of such a field pattern (for simplicity calculated in Cartesian coordinates since exp and cos are better known and easier to manipulate than J_m and I_m, especially with complex arguments), perfectly complying with Maxwell's equations is shown in detail in Appendix A2. It has the E_z component (all parameters real)

$$E_z = \exp(-\beta\cdot z - \delta\cdot x)\cos(\alpha\cdot z + \gamma\cdot x)\cdot\cos(\omega\cdot t) \quad (5)$$

and evidently cannot be expressed (for β≠0) as any combination of traveling waves due to the exponential exp(-β·z) factor (real β).

For x=0, as an 'on axis' example, the E_z field becomes

$$E_z = \exp(-\beta\cdot z)\cdot\cos(\alpha\cdot z)\cdot\cos(\omega\cdot t) \qquad (6)$$

Then a relativistic particle moving into the positive infinite half-space from z=0 to z=+∞ as z=c·t feels an integrated force proportional to

$$f_{\|} = \int_0^{+\infty} \exp(-\beta\cdot z)\cos(\alpha\cdot z)\cos(\omega z/c)dz \quad (7a)$$

identical to

$$f_{\|} = \frac{\beta/2}{\beta^2 + (\alpha-\omega/c)^2} + \frac{\beta/2}{\beta^2 + (\alpha+\omega/c)^2} \quad (7b)$$

Obviously for β≠0 (see later) $f_{\|}$ is *never equal to zero*. It has a maximum for (not precisely) α=ω/c, i.e. when the oscillatory part is (about) synchronous[h]. But it is also interesting to observe that *all* non-synchronous parameter configurations contribute; the farther off from synchronism they are, the less they contribute. This means that the "averaging away" of non-synchronous components does not work under *most* conditions, only for β=0. If δ=0 at the same time, it is a 'classical' traveling wave and then β→0, forcing α→ω/c due to the frequency constraint. In addition $f_{\|}$ becomes a delta-function, infinitely high at α=±ω/c but also infinitely sharp. For β=0 the above field-pattern has 'degenerated' into a 'classical' standing wave that can then be 'decomposed' into two traveling waves, which cannot be done for any field with β≠0.

This shows that 'classical' traveling waves are only a very special limiting case of the general set of solutions.

A very important contribution from the 'evanescent modes' was seen in the disk-loaded wave-guide. There the direct contribution to acceleration may be small, but they allow phase corrections to the main wave. 'Portions' of a non-synchronous wave can accelerate relatively efficiently over a short (cell-) length; hence synchronicity is locally not very essential. The phase of the main wave is then 'readjusted' over a small distance 'around each obstacle' (the iris) so that on average the overall field always appears synchronous without actually being 'the' synchronous wave, the latter requiring constant phase and amplitude from z=–∞ to z=+∞. Such a phase-shift cannot be realized by pure traveling waves alone. To fulfill Maxwell's equations 'evanescent modes' have to be mixed in around the obstacle.

The same is true for an isolated cavity on an infinite beam tube. 'Evanescent modes' allow part of a double traveling (i.e. a standing) wave in the main cavity volume to be "clipped off" while respecting Maxwell's equations. Then the integral over E_z along somewhat more than the cavity length determines the full accelerating voltage and the question as to what the behavior of a hypothetical wave far away has to do with the acceleration, needs not be asked anymore.

VII. Analysis of some 'proofs'

The counter-example in section IV is in principle sufficient to demonstrate that any 'proof 'of (1) must be flawed in one way or the other. But it is more satisfactory to determine the exact location of the weak points in the chain of arguments.

Therefore 'proofs' for (1) or related statements that have come to the attention of the author of this paper have been examined. It could be shown that they are *all* not generally valid *for structures with non-constant*

[h] There is another 'synchronous' field for α=-ω/c, i.e. for a particle moving with c in opposite direction

cross-sections. The following list of 'proofs' is probably not exhaustive, but it is expected that others would go along similar lines of thought. In any case the counter-example given above also guarantees that others cannot hold.

One consideration is important and might have already been said in the introduction. Since nature does not supply a charged particle without mass, due to relativity no electrically accelerated particle can ever reach v=c. Therefore any physical relation has to be demonstrated for v<c first, including its Lorentz invariance, e.g. when allowing an observer to sit on the trailing particle. If then there exists a relation such that the precise v<c cases converge as v→c, it can be called 'the v=c solution'. Then it is unimportant if, say, the particle is 10^{-5}·c or 10^{-6}·c away from c, the 'v=c solution' is precise enough. However, demonstrating relations by simply fixing β=1 does not allow Lorentz invariance to be verified and convergence as v→c for the proposed solution cannot be demonstrated - there might be divergence.

VII.1 V_{acc} calculation in cavities

Probably the oldest paper following this line of thought seems to be from 1983 [4]. There the intention in the cavity field-calculations is to avoid integration along large lengths of the cut-off tube with low field when calculating the on-axis V_{acc} (or only slightly off axis for m > 0) by using the fact that E_z=0 on the cut-off tube wall and only integrating over the actual cavity part at this cut-off radius. Then, exploiting the supposed, simple, r^m dependence of these longitudinal integrals, yields the desired results.

But in the appendix of [4] the r^m dependence of mode m is 'derived' by supposing that *only* contributions from traveling waves with k(ω)βc=ω in the asymptotic limit as γ→∞, (β→1), i.e. synchronous waves, are needed.

This means that, especially in cavities where fringe-fields exist at cavity-tube transitions, the essential evanescent modes are excluded thus imposing r^m 'by definition'. Therefore this method is not correct and should not be applied for precise results.

Then also the results derived in papers such as [12], which assume the radial dependence (citing [4]), seem questionable since elaborate calculations are based on wrong fundamental assumptions.

VII.2 Using 'Generalized Integration Contour'

In [3] it is claimed[i] that using a method called the 'Generalized Integration Contour' it can be shown that for the m=0 mode the longitudinal energy gain is independent of the radial coordinate where the integration was done.

For the reader's convenience the essential part of the chain of arguments as given in [3] is repeated and a few comments are added on the way.

The longitudinal 'induced voltage per driving charge q' is

$$G_{\|}(r,s) = -\frac{1}{q}\int_{-\infty}^{+\infty} dz \;\; E_z(r,z,t=(s+z)/v) \qquad (8)$$

All fields are the superposition of a drive field ('leading particle') and *radiated fields*; one considers here only the latter field (the leading particle with its 'field-disk' is gone when the trailing one arrives). Maxwell's (vacuum) equations yield[j]

$$curl_\theta(E) = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t} \quad \Rightarrow \frac{\partial E_z}{\partial r} = \frac{\partial E_r}{\partial z} + \frac{\partial B_\theta}{\partial t} \qquad (9)$$

From the definition of the 'total derivative'

$$\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} \qquad (10)$$

in the original text [3], equation (12) is found directly. Here some intermediate steps in this reasoning are added. Equation (10) is applied as

$$\frac{\partial E_r}{\partial z} = \frac{dE_r}{dz} - \frac{1}{v}\frac{\partial E_r}{\partial t} \qquad (11a)$$

$$\frac{\partial B_\theta}{\partial z} = \frac{dB_\theta}{dz} - \frac{1}{v}\frac{\partial B_\theta}{\partial t} \quad \Rightarrow \quad \frac{\partial B_\theta}{\partial t} = v\frac{dB_\theta}{dz} - v\frac{\partial B_\theta}{\partial z} \qquad (11b)$$

and is injected into (9) to yield

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}(E_r + vB_\theta) - \frac{1}{v}\frac{\partial E_r}{\partial t} - v\frac{\partial B_\theta}{\partial z} \qquad (12)$$

In the original text [3] from Maxwell's (vacuum) equation,

$$curl_r(B) = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = \varepsilon_0\mu_0\frac{\partial E_r}{\partial t} = \frac{1}{c^2}\frac{\partial E_r}{\partial t} \qquad (13)$$

and without any further comment

$$\frac{\partial E_r}{\partial t} \;=\; -c^2 \cdot \frac{\partial B_\theta}{\partial z} \qquad (14)$$

is concluded; hence implicitly it is assumed that

$$\frac{\partial B_z}{\partial \theta} \;\equiv\; 0 \qquad (15)$$

This is in fact justified for m=0 since all field components, including B_z, are independent of θ.

Injecting (14) into (12) yields

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}(E_r + vB_\theta) - \frac{1}{v}\frac{\partial E_r}{\partial t}(1 - \frac{v^2}{c^2}) \qquad (16a)$$

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}(E_r + vB_\theta) - \frac{1}{v\gamma^2}\frac{\partial E_r}{\partial t} \qquad (16b)$$

For ultra-relativistic speeds v→c, γ is very large;

[i] Appendix 3.A, part (3.A.1)

[j] As usual (r,θ,z) is defined right-handed: $f_r=e(E_r\text{-}v\cdot B_\theta)$, $curl_\theta(A)=dA_r/dz–dA_z/dr$, and $curl_r(A)=1/r\cdot dA_z/d\theta–dA_\theta/dz$

hence the $1/\gamma^2$-term can be neglected, i.e.

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}(E_r + vB_\theta) \tag{17}$$

and then

$$\frac{\partial}{\partial r}G_{\|}(r,s) = -\frac{1}{q}\int_{-\infty}^{+\infty} dz\ \frac{\partial E_z}{\partial r}(r,z,t=(s+z)/v) = -\frac{1}{q}(E_r + vB_\theta)\,|_{z=-\infty}^{z=+\infty} \tag{18}$$

Closing argument: the radiated fields must vanish somewhere far away, imposing $\partial G/\partial r = 0$ and hence the longitudinal force is independent of the radial position,

$$\frac{\partial}{\partial r}G_{\|}(r,s)\ =\ 0\ \Rightarrow\ G_{\|}(r,s)\ =\ const(s) \tag{19}$$

which finishes the text in [3] concerning the case m=0.

But in the following parts of [3] other derivations are made that stand and fall by the validity of (18) and (19).

It can be seen very rapidly that the above 'proof' cannot be correct: (15) is a necessary condition that seems to limit the validity to m=0 modes only. However, all TM modes (as the name Transverse Magnetic expresses) have $B_z \equiv 0$ and as a consequence $\partial B_z / d\theta \equiv 0$, hence the 'proof' as presented will not only pass for m=0 modes but for *all* fields with an integrated beam interaction. TE modes are irrelevant, having no integrated beam interaction. Then the chain of arguments given above would prove that *all* longitudinal wakes are independent of r, in clear contradiction even to statements made elsewhere in the same book [3]. Also according to PW, there could never be an electromagnetic field causing a transverse momentum kick, clearly in disagreement with operational RF particle separators.

In fact, equation (16b) is perfectly correct. However, it is not justified to neglect the second right-hand term for v→c. This term is a product of two factors, $1/\gamma^2$ and dE_r/dt. But, as will be shown in the following, the second factor generally scales as γ^2 when v→c so that the product remains asymptotically constant and cannot be neglected as claimed in [3].

To show this, the same equations as in [3] – reproduced in (9) to (19) – are used. Combining (9) and (11a) yields

$$\frac{\partial E_z}{\partial r} - \frac{dE_r}{dz} = -\frac{1}{v}\frac{\partial E_r}{\partial t} + \frac{\partial B_\theta}{\partial t} \tag{20}$$

It can be rearranged and when d/dt is expressed as v·d/ds, one obtains

$$\frac{\partial E_z}{\partial r} = \frac{dE_r}{dz} - \frac{\partial}{\partial s}\left(E_r - v\cdot B_\theta\right) \tag{21}$$

which is in fact nothing else than the PW theorem, $f_r/e = E_r - v\cdot B_\theta$ being the transverse Lorentz force per charge (*all* modes included, TM *and* TE). Then by integrating over (the 'absolute') z and considering that fields 'disappear far away', this results in

$$\frac{\partial}{\partial r}G_{\|} = \frac{1}{q}E_z\Big|_{-\infty}^{-\infty} - \frac{1}{qe}\frac{\partial}{\partial s}\int_{-\infty}^{+\infty} f_r dz = -\frac{1}{qe}\frac{\partial F_r}{\partial s} \neq 0 \tag{22}$$

Therefore in general

$$\frac{\partial}{\partial r}G_{\|}(r,s)\ \neq\ 0\ \Rightarrow\ G_{\|}(r,s)\ \neq\ const(s) \tag{23}$$

disproving (19) and with it the results of the 'generalized integration contour' method.

The erroneous conclusion that (19) is always zero comes from the fact that the ratio of transverse to longitudinal fields increases as γ^2 for fields that have an integrated beam interaction. This can be seen as follows: Combining (14) – hence considering only TM modes, TE having no beam interaction – and (11b) one gets

$$-v\frac{dB_\theta}{dz}\ =\ \frac{v}{c^2}\frac{\partial E_r}{\partial t} - \frac{\partial B_\theta}{\partial t} \tag{24}$$

Now one can add chosen multiples of (20) and (24). Simply adding (20) and (24) cancels the B_θ time derivative term yielding precisely (16b), i.e.

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}(E_r + vB_\theta) - \frac{1}{v\gamma^2}\frac{\partial E_r}{\partial t} \tag{25}$$

Adding (20) and c^2/v^2 times (24) cancels the E_r time derivative term yielding

$$\frac{\partial E_z}{\partial r} = \frac{d}{dz}\left(E_r + \frac{c^2}{v}B_\theta\right) - \frac{c^2}{v^2\gamma^2}\frac{\partial B_\theta}{\partial t} \tag{26}$$

and in both cases one is tempted to the same wrong conclusion that the second right hand term disappears for v→c, hence γ→∞.

Equations (21), (25) and (26) all have the same left hand side and on the right hand side in all three cases an absolute derivative, with respect to z, of a linear combination of E and B, these fields being assumed to disappear 'far away'. Then the infinite integral over the three remaining right hand terms must be equal

$$\int_{-\infty}^{+\infty} dz\frac{\partial}{\partial t}\left(E_r - v\cdot B_\theta\right) = \frac{1}{\gamma^2}\int_{-\infty}^{+\infty} dz\frac{\partial E_r}{\partial t} = \frac{1}{\gamma^2}\frac{c^2}{v}\int_{-\infty}^{+\infty} dz\frac{\partial B_\theta}{\partial t} \tag{27}$$

hence the transverse fields E_r and B_θ scale (on average) as γ^2. For $E_r - v\cdot B_\theta$, which gives the deflecting Lorentz force, both contributions mutually cancel more and more in such a manner that the total integrated deflection asymptotically approaches a constant.

This also shows up in a simple example with traveling waves with phase-velocity v while v→c. In Cartesian coordinates the usual traveling wave solutions are presented as products of factors of angular functions (sin or cos) of $k_x x$, $k_y y$ and $\omega t - k_z z$. Since div(E)=0, the amplitudes of TM modes with unit amplitude for the longitudinal E_z-field (TE has $E_z=0$) behave as

$$E_x\ \propto\ -k_x k_z/(k_x^2 + k_y^2) \tag{28a}$$

$$E_y \propto -k_y k_z/(k_x^2+k_y^2) \quad (28b)$$
$$E_z \propto 1 \quad (28c)$$
$$B_x \propto -k_y\omega/(k_x^2+k_y^2)/c^2 \quad (29a)$$
$$B_y \propto -k_x\omega/(k_x^2+k_y^2)/c^2 \quad (29a)$$
$$B_z = 0 \quad (29c)$$

and the specific Lorentz force $f_x/e=(E_x-vB_y)$ becomes

$$f_x/e = E_x - vB_y \propto (v\omega/c^2-k_z)k_x/(k_x^2+k_y^2) \quad (30)$$

(similar for f_y/e). The frequency constraint equivalent to (4) in Cartesian coordinates is

$$(\omega/c)^2 = k_x^2 + k_y^2 + k_z^2 \quad (31)$$

Equation (31) allows the denominators above to be expressed by k_z and ω and enforcing the phase velocity v to be $k_z=\omega/v$ one gets

$$\frac{-1}{k_x^2+k_y^2}=\frac{-1}{\omega^2/c^2-k_z^2}=\frac{v^2/\omega^2}{1-v^2/c^2}=\gamma^2\frac{v^2}{\omega^2} \quad (32)$$

and

$$\frac{v\omega/c^2-k_z}{k_x^2+k_y^2}=-\gamma^2\frac{v}{\omega}(1-v^2/c^2)=-\frac{v}{\omega} \quad (33)$$

Then it becomes clear that all transverse fields increase as γ^2 as v→c while the Lorentz force remains constant.

This scaling is not in contradiction to the relativistic transformation laws where the longitudinal fields (in the direction of movement) remain constant while transverse fields scale as γ, not γ^2. In fact the relativistic transformation considers a wave to have frozen properties when seen from different moving systems while a wave with changing (phase velocity) conditions is observed from the fixed laboratory system.

There is one problem for these traveling waves with v<c: k_x and k_y have to be imaginary to fulfill the frequency constraint (30) and fields (including E_z) permanently grow in the transverse direction, hence there is no zero in E_z possible on a closed tube boundary. These are not 'classical' traveling waves which can be used on their own in a closed tube but only partial solutions for field matching.

VII.3 Using curl(F) and div(F)

In [2], chapter 1.2 and 1.3, a chain of arguments is given to demonstrate the general r^m dependence of the m^{th} multipole. Here the demonstration in [2] is essentially repeated but a few comments are given on the way.

The local Lorentz force for a particle with speed βc in the z-direction is

$$\vec{f} = e\cdot\left(\vec{E} + \beta\cdot c\cdot\hat{z}\times\vec{B}\right) \quad (34)$$

One can calculate curl and div of f by exploiting Maxwell's vacuum equations (a driving charge is gone when the trailing particle arrives), resulting in

$$curl(\vec{f}) = -e\cdot\left(\frac{1}{c}\frac{\partial}{\partial t} + \beta\cdot\frac{\partial}{\partial z}\right)\vec{B} \quad (35)$$

$$div(\vec{f}) = -\frac{e\cdot\beta}{c}\frac{\partial E_z}{\partial t} \quad (36)$$

The momentum kick for such a (trailing) particle a distance D behind the leading particle is

$$\Delta\vec{p}(x,y,D) = \int_{-\infty}^{+\infty} dt\cdot f(x,y,D+\beta ct,t) \quad (37)$$

Taking the curl of (37) while expressing dz by dD, exchanging integration and curl and exploiting (35) leads to

$$curl_D(\Delta\vec{p}(x,y,D)) = -e\int_{-\infty}^{+\infty} dt\cdot\left[\left(\frac{1}{c}\frac{\partial}{\partial t} + \beta\cdot\frac{\partial}{\partial z}\right)\vec{B}(x,y,z,t)\right]_{z=D+\beta ct} \quad (38)$$

where the $curl_D$ operator takes differentiation with respect to D instead of z as for the standard curl. Equation (38) can be integrated immediately as

$$curl_D(\Delta\vec{p}(x,y,D)) = -\frac{e}{c}\vec{B}(x,y,D+\beta ct,t)\Big|_{t=-\infty}^{t=+\infty} \quad (39)$$

The right hand side either absolutely vanishes, e.g. for an isolated cavity, or for a periodic structure it remains bound while Δp increases proportionally to the integration path, i.e. the right hand side relatively vanishes. This results in any case in the statement

$$curl_D(\Delta\vec{p}(x,y,D)) = 0 \quad (40)$$

In fact this is a – perhaps unfamiliar – way to write the PW theorem. As is well known, it is exactly *valid for all v*, for any v as close as desired to c.

In components, writing the integrated forces F_r instead of Δp_r (z and θ similar) in cylindrical coordinates, this means

$$\frac{1}{r}\frac{\partial F_z}{\partial\Theta}-\frac{\partial F_\Theta}{\partial D} = 0 \quad (41a)$$

$$\frac{\partial F_r}{\partial D}-\frac{\partial F_z}{\partial r} = 0 \quad (41b)$$

$$\frac{1}{r}\left(\frac{\partial(rF_\Theta)}{\partial r}-\frac{\partial F_r}{\partial\Theta}\right) = 0 \quad (41c)$$

This system does not yet impose any power-of-r dependence. In fact one can see that for any given azimuthal mode number m, a partial solution with *any* j ≥ m, in particular j≠m, as

$$F_z = r^j\cdot W_j'(D)\cdot\cos(m\cdot\theta) \quad (42a)$$

$$F_r = j\cdot r^{j-1}\cdot W_j(D)\cdot\cos(m\cdot\theta) \quad (42b)$$

$$F_\theta = -m\cdot r^{j-1}\cdot W_j(D)\cdot\sin(m\cdot\theta) \quad (42c)$$

fulfills (41). Therefore for a given m many terms with different j can be superimposed – e.g. to form a Bessel function as $J_m(k_r r)$ – still fulfilling the curl(F)=0 condition, i.e. PW.

Explicitly in cylindrical coordinates, (36) becomes

$$\frac{1}{r}\frac{\partial(rf_r)}{\partial r}+\frac{1}{r}\frac{\partial f_\Theta}{\partial\Theta}+\frac{\partial f_z}{\partial z}= div(f) = -\frac{e\beta}{c}\frac{\partial E_z}{\partial t} \quad (43)$$

In the original text [2] β is immediately set equal to 1 but it is important to keep it a free variable. One can calculate the integrated forces and replacing f_z by its definition $e\cdot E_z$ yields

$$\frac{1}{r}\frac{\partial(rF_r)}{\partial r}+\frac{1}{r}\frac{\partial F_\Theta}{\partial\Theta} = -e\int_{-\infty}^{+\infty} dt\cdot\left[\left(\frac{\beta}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)E_z\right]_{z=D+\beta ct} \quad (44)$$

Here [2] concludes – with β immediately replaced by 1, fixing $v\equiv c$ from the beginning – that (see exercise 2, middle line of eq. 1.18) the right hand side of (44) would *always* be zero. If this would be true, (44) together with (42) would in fact impose $j^2=m^2$ and with it the claimed canonical configuration (1).

The right hand side of (44) disappears asymptotically for *traveling waves with phase velocity identical to v=β·c*, i.e. fields proportional to cos(ωt–ωz/(β·c)). Under these conditions where z and t enter as the common parameter t-z/(β·c), the integrand of (44) is proportional to

$$\frac{\beta\cdot\omega}{c}-\frac{\omega}{\beta\cdot c}=\frac{\omega}{\beta c}(1-\beta)^2=\frac{\omega}{\beta c}\frac{1}{\gamma^2} \quad (45)$$

which then tends to zero as $1/\gamma^2$ while v→c.

However, it was already stated before that the deviations from the canonical form (1) are due to the non-traveling wave fields necessarily present in non-smooth structures where t and z do not enter in the above manner.

In the integral (38) β is the factor of the derivative *with respect to* z and therefore the absolute derivative of B(x,y,z=D+βct,t) is identical to the integrand of (38). Intrinsically this respects the particle movement constraint z=D+βct and hence the above B is a primitive function. Then the integral is the difference of this B at the limit of the integration range and since fields can be assumed to disappear far away, the integral is in fact equal to zero. This (i.e. PW) is precisely – not only asymptotically for v→c – the case for *any* Maxwellian field and *any* velocity v=βc.

In contrast to this, in (44) β is the factor of the derivative *with respect to t* and the same argument cannot be used anymore: E(x,y,z=D+βct,t) is *not* a primitive function with respect to t. The function β/c·E(x,y,z=D+c/βt,t) seems to be a primitive function but it does not respect the necessary movement constraint z=D+βct. Thus, even if E vanishes at the end of the integration range this does not tell us anything about the integral in (44). When z and t enter as completely separate variables, not as the common parameter t-z/(β·c) as for traveling waves, nothing can be concluded on the asymptotic behavior of the integral for v→c.

Therefore the chain of arguments presented in [2] does not conclusively show that for all Maxwellian fields (44) is – or for v→c tends towards – zero and hence the canonical configuration (1) in non-smooth structures was not proven.

VIII. The difference

Once the non-validity of (1) is accepted, it is necessary to study how the ‘new’ reality compares to the ‘old’ one, are there large differences and if yes, where.

The configuration (1) is restricted to only the lowest order term in r, i.e. r^m for the m^{th} multipole, claiming that all higher terms are *in principal* absent. Since it is evident now that these terms are generally not absent, one can use (1) *close to the axis* as an *approximation* to the true longitudinal integrated wake-force. This is even better than one could hope for, under the circumstances: Bessel-Functions have only non-zero coefficients each *second* power. Therefore (1) can be considered as a 1^{st} order approximation, neglecting the 2^{nd} and higher order coefficients.

However, there is one point of concern. In (1) the 2^{nd} order term of the monopole mode was considered to be always exactly zero. For the longitudinal wakes this will generally not make a big difference; but things are different for the transverse integrated forces.

PW states that the radial momentum kick is proportional to the derivative of the longitudinal forces with respect to the radial position. Since each monopole term has a true parabolic shape as $1-\alpha\cdot r^2$, this creates a transverse momentum kick proportional to 2α·r; the constant (main) longitudinal term is ‘differentiated away’. In (1) the coefficient α was forced to be zero but in reality resulting statements such as: “in a round structure an on-axis beam does not excite transverse momentum kick (even) for off-axis trailing particles” *do not hold anymore*.

Also for the so-called ‘quadrupolar wakes’ it is assumed that with certain geometrical symmetries (e.g. up-down or left-right symmetries) and an x-kick proportional to the x-deviation with a proportionality constant D, i.e. Δp_x = D·x, then the y-kick is proportional to the same constant with opposite sign. i.e. Δp_y = –D·y. In reality the 2^{nd} order term of the monopole wake adds to this, resulting in $\Delta p_x=(2\alpha+D)\cdot x$ and $\Delta p_y=(2\alpha-D)\cdot x$, hence the supposed symmetry configuration of the wakes is not true and may lead to wrong results when enforcing it by a constrained fit.

Finally, also the main LHC RF cavities, to very good approximation round, will produce a radial kick for off-axis particles. This is also true for monopole HOMs in these cavities.

α is generally not very large, a coarse estimate is that in a cavity of radius R only $1\text{-}\alpha \cdot R^2$ becomes zero (for the lowest modes). Therefore in a numerical calculation the effect may be difficult to discover from *longitudinal forces* when applying PW and only testing with small offsets r. However, by direct integration of the radial forces from E_r and $v \cdot B_\theta$ the numerical problem of the 'difference of large numbers' will disappear and the radial kick will become clearly visible.

For dipole wakes the first 'new' term is proportional to r^3 and its derivative with respect to r proportional to r^2. Therefore these effects are already in the non-linear range and are often neglected. However a treatment taking non-linear terms into account should also include these.

IX. Conclusions

It has been shown that there is an infinity of field patterns that have non-zero longitudinal integrated force, each one having its own configuration in (r,θ). Therefore the claim that only 'the' synchronous wave can accelerate v_p=c particles, hence imposing its unique (r,θ) behavior, is vain. The existence of a large number of modes, not accounted for in (1), capable of producing integrated wakes was shown.

Finally any other description which always excludes these other evanescent field patterns cannot exist: a clear counter example, contradicting (2), and with it (1), was shown.

Last, but not least, several known 'proofs' for (1) were shown to be erroneous for non-smooth structures,

Therefore the radial configuration of the true wakes can only be determined by considering all details of the boundary and is not equal to a general function independent of these details.

This means that the configuration (1) is too rigid and should be replaced by the softer conditions

$$\int_{-\infty}^{+\infty} ds \cdot \vec{F}_{\|}^{(m)} = -e \cdot cos(m\Theta) \cdot I_m \sum_{n=0}^{\infty} W'_{m,n}(z) \cdot r^{m+2n} \quad (46a)$$

$$\int_{-\infty}^{\infty} ds\, \vec{F}_{\perp}^{(m)} = -eI_m \cdot \sum_{n=0}^{\infty} (m+2n) \cdot W_{m,n}(z) \cdot r^{m+2n-1} \cdot (\ddot{\rho} \cdot cos(m\Theta) - \ddot{\Theta} \cdot sin(m\Theta)) \quad (46b)$$

exploiting the fact that $J_m(x)$ has the lowest power x^m and a non-zero coefficient for only each second power.

The multipoles m>2 are normally neglected, the second non-zero term for m=1 is proportional r^3, i.e. delivers r^2-terms for the transverse force, which is generally also neglected.

The essential difference between (1) and the corrected form (46) is therefore the r^2 term of the monopole mode. This means that, in contrast to (1), in reality monopoles also create a transverse momentum kick proportional to r, absent in (1), not only the quadropole wakes (lowest term for m=2).

Therefore all conclusions based on the configuration (1) should be revised and recalculated, respecting the true actual boundary conditions.

It should also be considered that the numerically simplified integration of V_{acc} at the level of the cut-off tube radius of a cavity is not truly identical to the – correct – axial integration, even if the difference is expected to be small.

Furthermore, the application of (1) to constrain wake components in the data analysis of impedance measurements of objects should be revised.

Practically for LHC this means, amongst others, that round objects with longitudinally changing cross-section (any diameter changes in the tube, even if cylindrically symmetric) can also produce transverse momentum kicks for off-axis trailing particles. The same is true for the round accelerating RF cavities.

This may not necessarily all be bad, e.g. transverse Landau damping might be increased, but the effects should be verified considering the true full component geometry.

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Appendix

Several textbooks on electro-magnetism and RF show the solutions to Maxwell's equation in cylindrical co-ordinates, considerations on plane waves, field matching and Bessel-functions in exact detail (see e.g. [6][7][8][13][14][15]). For the reader's convenience some of these facts have been compiled in this appendix. The style is kept heuristic.

A1. Solutions of Maxwell's equations in cylinder coordinates

In all accelerator problems there is a special axis, the beam direction. Therefore, the chosen co-ordinate system should reflect this. Furthermore, round objects are often important and hence a cylindrical coordinate system is the obvious (but not necessary) choice.

The wave equation (in vacuum) appears as a double application of the curl differential operator on a chosen field component (either E or H), i.e.

$$curl(curl(\vec{E})) = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{(A1)}$$

Generally the double curl operator contains many mixed derivatives. Since in vacuum div(E)=0, one may subtract grad(div(E)) from (A1) yielding a generally simpler representation

$$\Delta E = curl(curl(\vec{E})) - grad(div(\vec{E})) = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{(A2)}$$

To have a truly physical solution it must always be guaranteed separately that div(E)=0 holds.

Writing (A2) in cylindrical components one obtains for E_z alone the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \quad \text{(A3)}$$

E_r and E_θ are mixed in the two equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial r}\right) - \frac{E_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 E_r}{\partial \theta^2} + \frac{\partial^2 E_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial E_\theta}{\partial \theta} = \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} \quad \text{(A4)}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_\theta}{\partial r}\right) - \frac{E_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 E_\theta}{\partial \theta^2} + \frac{\partial^2 E_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial \theta} = \frac{1}{c^2}\frac{\partial^2 E_\theta}{\partial t^2} \quad \text{(A5)}$$

(A3) is solved in the standard way, writing E_z as the product of four independent functions of the four variables r, θ, z and t. As in Cartesian coordinates the t and z-dependence are written as cos(ωt) and $\cos(k_z z)$ – where cos → sin yields three more sets of linearly independent solutions. The θ-dependence is expressed as cos(mθ) (sin(mθ) yielding another set of linearly independent functions) and due to the 2π-symmetry of space m has to be integer. With the, as yet unknown, radial function f(r) and the amplitude A_z this yields the expression

$$E_z = A_z \cdot f(r)\cdot\cos(m\theta)\cdot\cos(k_z z)\cdot\cos(\omega\cdot t) \quad \text{(A6)}$$

Injecting (A6) into (A3) results in

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right) + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2} - k_z^2\right) f(r) = 0 \quad \text{(A7)}$$

where the common, globally non-zero, factor $A_z\cos(m\theta)\cos(k_z z)\cos(\omega t)$ has been left out. Writing f(r) as $y(x{=}k_r r)$ with an as yet undefined parameter k_r yields

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x}\frac{\partial y(x)}{\partial x} + \left(\frac{(\omega/c)^2 - k_z^2}{k_r^2} - \frac{m^2}{x^2}\right) y(x) = 0 \quad \text{(A8)}$$

If the first part in the large bracket is equal to 1, i.e.

$$k_r^2 + k_z^2 = (\omega/c)^2 \quad \text{(A9)}$$

(A8) becomes the well-known Bessel differential equation

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x}\frac{\partial y(x)}{\partial x} + \left(1 - \frac{m^2}{x^2}\right) y(x) = 0 \quad \text{(A10)}$$

with solutions given by the two independent functions $J_m(x)$ and $Y_m(x)$. Y-type solutions will be considered later. The fields are then a linear combination of solutions as (leaving away cos(ω t))

$$E_z = A_z \cdot J_m(k_r r)\cdot\cos(m\theta)\cdot\cos(k_z z) \quad \text{(A11)}$$

where the frequency constraint (A9) always has to be respected.

Without making explicit calculations, the corresponding solutions of (A2) and (A3) are of the type

$$E_r = -\ddot{A}\cdot J'_m(k_r r)\cdot\cos(m\theta)\cdot\sin(k_z z) \quad \text{(A12)}$$

$$E_\theta = \ddot{A}\cdot(m/r)J_m(k_r r)\cdot\sin(m\theta)\cdot\sin(k_z z) \quad \text{(A13)}$$

plus another independent set of the type

$$E_r = \ddot{C}\cdot(m/r)J_m(k_r r)\cdot\cos(m\theta)\cdot\sin(k_z z) \quad \text{(A14)}$$

$$E_\theta = -\ddot{C}\cdot J'_m(k_r r)\cdot\sin(m\theta)\cdot\sin(k_z z) \quad \text{(A15)}$$

The amplitude coefficients are not independent, the condition div(E) has to be fulfilled. Therefore there exist only two linearly independent 3D solutions. It is a convention to express these solutions in a standard form where one solution has no E_z-field (C coefficients in (A14) and (A15)) called TE-mode (Transverse Electric) or H-mode. These, lacking E_z, do not interact longitudinally with the beam and then from PW also do not interact transversely.

The other combination has no longitudinal H field, and is called the TM-mode (Transverse Magnetic) or E-mode. (A-coefficients in (A11),(A12) and (A13)). Applying the condition div(E)=0 one gets the TM mode solutions with a common amplitude factor A - all oscillating as cos(ωt).

$$E_r = -A\cdot k_z\cdot k_r\cdot J'_m(k_r r)\cdot\cos(m\theta)\cdot\sin(k_z z) \quad \text{(A16)}$$

$$E_\theta = A\cdot k_z\cdot(m/r)J_m(k_r r)\cdot\sin(m\theta)\cdot\sin(k_z z) \quad \text{(A17)}$$

$$E_z = A\cdot k_r^2\cdot J_m(k_r r)\cdot\cos(m\theta)\cdot\cos(k_z z) \quad \text{(A18)}$$

A completely equivalent set exists with Y_m instead of J_m (the linear combinations J±i·Y are called Hankel-functions). The functions Y_m all diverge at the axis r=0; their coefficient is zero for free waves in vacuum in situations lacking a central conductor. With a central conductor, a linear combination of J_m and Y_m can be arranged such that for all z and θ (in a smooth pipe) E_z is zero as well on the inner as on the outer conductor. The circular coaxial line, technically very important as the coaxial cable, carries the limiting so-called TEM mode, i.e. E and H perpendicular to the propagation, as in a light ray. Then the coefficients of J_0 and J_0' are zero, as well as the one of Y_0, describing E_z, while the only remaining Y_0'-term, describing E_r, 'degenerates asymptotically' to become 1/r. Exceptionally, the phase and group velocity are equal to c for this mode (if no filling material with ε>1 is used).

The solutions described above have a spatial field map E(r,θ,z) that is oscillating coherently with cos(ωt), i.e. at one instant the field is maximum everywhere, $T_{osc}/4$ later it is zero everywhere (and the H-field is maximum), a standing wave. The double factors

$$\cos(k_z z)\cdot\cos(\omega\cdot t) \qquad \text{(A19)}$$

can formally be written as a linear combination of

$$\cos(\omega\cdot t \pm k_z z) \qquad \text{(A20)}$$

and solutions appear as the superposition of traveling waves, formally written as

$$E_z = A_z\cdot f(r)\cdot\cos(m\theta)\cdot\cos(\omega\cdot t - k_z z) \qquad \text{(A21)}$$

It seems that the same field pattern can be found Δt later at a location shifted by $\Delta z=\omega/k_z\Delta t$, i.e. the wave seems to move with the (phase) velocity $v=\omega/k_z$. However, for non-decaying solutions ω is real and this interpretation of (A19) as a traveling wave only holds for *real k_z, otherwise there is no traveling wave*!

An essential point is that for all solutions k_r, k_z and ω are constrained by (A9) (identical to (4)) in order to be solution to Maxwell's equations. Any solution for a set (k_r,k_z,ω) is linearly independent from all the other ones.

The solutions with real (k_r,k_z,ω) can describe all waves in smooth tubes. Therefore complex parameters are often neglected, but for non-smooth tubes the full solutions are essential.

For non-decaying solutions ω has to remain real. J and cos can be presented as a power series convergent for any argument, hence they are also defined for complex arguments. Injecting the complex parameter k e.g. in (A2), creates a complex field. When the frequency constraint (A5) is respected, this complex field reproduces itself with a (real) scaling factor $(\omega/c)^2$. Then the real and imaginary part are two new independent solutions of Maxwell's equations in cylindrical coordinates. But, as shown above, for non-real k_z they cannot be 'split' into traveling waves as is the case for the solutions with purely real k_z.

For complex k_r and k_z but real ω, the constraint becomes

$$k_{r,\mathrm{Re}}\cdot k_{r,\mathrm{Im}} + k_{z,\mathrm{Re}}\cdot k_{z,\mathrm{Im}} = 0 \qquad \text{(A22)}$$

$$k_{r,\mathrm{Re}}^2 - k_{r,\mathrm{Im}}^2 + k_{z,\mathrm{Re}}^2 - k_{z,\mathrm{Im}}^2 = (\omega/c)^2 \qquad \text{(A23)}$$

These solutions can be classified as the 'evanescent mode' as soon as k_z has a non-vanishing imaginary part.

The two limiting cases for purely real frequency ω will now be briefly discussed.

The first case has imaginary $k_r\neq 0$ but real k_z. The fact that k_z is purely real means that these solutions still have standing/traveling wave character. The frequency condition ($i\cdot k_r\rightarrow k_r'$) becomes

$$(\omega/c)^2 = k_z^2 - k_r'^2 \qquad \text{(A24)}$$

and $J_m(x)$ has to be replaced by its imaginary equivalent, the modified Bessel function $I_m(x)$ defined by

$$I_m(x) = i^{-m} J_m(i\cdot x) \qquad \text{(A25)}$$

I_m is always real. All I_m grow indefinitely with exponential character and have no zero except at x=0 for m>0. Hence they are not global solutions, valid everywhere, but can be used in the context of field matching [7][8]. For completeness, the imaginary equivalent of Y_m is the 2^{nd} order modified Bessel-function, usually written as K_m.

The second case uses imaginary k_z but real k_r. Then $\cos(k_z\cdot z)$ and $\sin(k_z\cdot z)$ become $\cosh(k_z\cdot z)$ and $\sinh(k_z\cdot z)$, that may be expressed by $\exp(\pm k_z\cdot z)$. The traveling wave character is lost for this type of solutions; fields are essentially confined in space where they oscillate coherently, i.e. at one instance there is maximum electric (zero magnetic) field everywhere, $T_{osc}/4$ later there is zero electric (maximum magnetic) field everywhere. The frequency condition ($i\cdot k_z\rightarrow k_z'$) becomes

$$(\omega/c)^2 = k_r^2 - k_z'^2 \qquad \text{(A26)}$$

These solutions disappear exponentially (normally very rapidly) for z→+∞ or z→–∞, hence the name 'evanescent modes'. But they also grow indefinitely in the opposite direction and hence also cannot be used as globally valid functions. In this role all solutions with $\mathrm{Im}(k_z)\neq 0$ are essential as local solutions superposed on the traveling waves for field matching [7][8], e.g. at tube diameter steps or irises, thus describing the 'fringe fields'

A2 Cartesian evanescent mode

In Cartesian coordinates the following field is a solution of Maxwell's equations. It can by no means be decomposed into traveling waves, except for β=δ=0.

$$E_z = \exp(-\beta\cdot z - \delta\cdot x)\cos(\alpha\cdot z + \gamma\cdot x) \qquad \text{(A27)}$$

$$E_y = 0 \qquad \text{(A28)}$$

$$E_x = \exp(-\beta\cdot z - \delta\cdot x)\times \left[A\cdot\sin(\alpha\cdot z + \gamma\cdot x + \psi) + B\cdot\cos(\alpha\cdot z + \gamma\cdot x + \psi)\right] \qquad \text{(A29)}$$

$$\alpha^2 - \beta^2 + \gamma^2 - \delta^2 = (\omega/c)^2 \qquad \text{(A30)}$$

$$\alpha\cdot\beta + \gamma\cdot\delta = 0 \qquad \text{(A31)}$$

$$A = \frac{\beta\cdot\gamma - \alpha\cdot\delta}{\gamma^2 + \delta^2};\; B = \frac{\beta\cdot\gamma + \alpha\cdot\delta}{\gamma^2 + \delta^2} \qquad \text{(A32)}$$

Using the angular function theorems, E_z can also be written as

$$E_z = \exp(-\delta \cdot x)\cos(\gamma \cdot x)\exp(-\beta \cdot z)\cos(\alpha \cdot z) - \exp(-\delta \cdot x)\sin(\gamma \cdot x)\exp(-\beta \cdot z)\sin(\alpha \cdot z) \quad \text{(A33)}$$

giving the field 'on axis' x=0

$$E_z(x=0) = \exp(-\beta \cdot z)\cos(\alpha \cdot z) \quad \text{(A34)}$$

(A30) and (A31) are the complex equivalent of (4) or (A9). A and B in (A32) are linked by div(E)=0. By replacing cos( ) by sin( ) in equations (A27) to (A33), the complementary solutions appear.

Only for β=δ=0 do these solutions 'degrade' to the 'classical' traveling waves. This shows conversely that for each of the latter an uncountable number of other solutions exist.

In cylindrical coordinates 'transverse products' such as $\exp(\delta_x x)\cdot\cos(\beta_x x)$ or $\exp(\delta_y y)\cdot\cos(\beta_y y)$ become Bessel functions of complex arguments as $J_m((\delta+i\cdot\beta)\cdot r)$ but the longitudinal products in z remain as above, i.e. the longitudinal force integrals keep the same configuration.

Bessel functions of complex arguments are not very well documented, only the purely imaginary case is known as the modified Bessel function I_m (see Appendix A1). Therefore the above derivation was done in Cartesian coordinates where exp and cos are well-known standard functions.

References

[1] Alexander Wu Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, (John Wiley & Sons, Inc., New York), 1993, Chap. 2, p.58.; out of print. An electronic version exists at http://www.slac.stanford.edu/~achao/wileybook.html

[2] Alex Chao, *Lecture Notes on Topics in Accelerator Physics*, March 2001, electronically available at http://www.slac.stanford.edu/~achao/lecturenotes.html

[3] B. W. Zotter and S. A. Kheifets, *Impedances and Wakes in High-Energy Accelerators*, World Scientific, Singapore, 1997

[4] T. Weiland, *Comment on wake field computation in time domain*, NIM (883), 31.

[5] W. K. H. Panofsky and W. A. Wenzel, Rev. Sci. Instr. **27**, 967 (856).

[6] Becker/Sauter, *Theorie der Elektrizität* (in German), (B. G. Teubner, Stuttgart), 1968, Vol. 1, p 183f

[7] R. E. Collin, *Field theory of guided waves*, IEEE Press, New York, 2nd ed., 1991, Chap. 8, p547 ff

[8] Gerhard Piefke: *Feldtheorie* (in German), (Bibliographisches Institut Mannheim, B.I. - Wissenschaftsverlag), 1977, Vol. 3

[9] Perry B. Wilson, **IX**th Conf. On High Energy Accel., Stanford, 1974, p. 57

[10] P. M. Lapostolle and A. L. Septier (ed.), *Linear accelerators*, North Holland Publishing Company, Amsterdam, 1970p. 729

[11] J.P. Pruiksma et al., *Electromagnetic fields in periodic linear traveling-wave structures*, Proc. 18th International Linear Accelerator Conference - Linac '96, 26 - 30 Aug 1996 , Geneva, Switzerland; ed. Charles Hill, Maurizio Vretenar http://www94.web.cern.ch/Linac96/Proceedings/Monday/MOP17/Paper.pdf

[12] O. Napoly, Y. H. Chin, B. Zotter, *A generalized method for calculating wake potentials*, NIM-A, Vol. 334, Iss. 2-3, 255 (1 Oct 1933)

[13] J. D. Jackson, *Classical Electrodynamics*, (John Wiley & Sons, New York), 2nd ed., 1975

[14] I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry*, (Oliver and Boyd, Edinburgh), 1961

[15] Jon Mathews, R. L. Walker, *Mathematical Methods of Physics*, W. A. Benjamin, Inc., New York), 2nd ed., 1970