AAS-Note 80
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15 August 1993

Effects of the Opening Angles to the Emittances[2]

The photons emitted by the synchrotron radiation are distributed transversely around the axis of the electron motion. This effect is treated in the envelope formalism. Radiation integrals are extended also. The effect is tiny for the damping rings of the future linear colliders.

1 Introduction

The synchrotron radiation has an angle with respect to the velocity of the emitting particle. This angle is called the opening angle. The effect of the vertical opening angle on the vertical emittance was roughly evaluated in Ref.[1]. In Ref.[2], it was done more carefully, taking into account the correlation between energy (γ) and the opening angle (ψ).

Recently, a paper[3] was published on the general formalism for calculating the equilibrium envelope matrix based on the envelope formalism. This formalism is suited particularly for a numerical calculation. Also, an extension of the radiation integrals were made. The opening angle, however, were not included, because the effect is known tiny. In this paper, we apply the envelope formalism to this effect just for the sake of the completeness. Although the effect is tiny for realistic ring parameters, present and near future, it is still interesting to do so and may be important for future extreme machines.

2 Classical Formulae

Power Spectrum Let us start with an equation in Ref.[4], which reads

$$\frac{dP}{d\omega d\phi d\psi} = \frac{\hbar\alpha}{4\pi^2}\omega^2 \int_{-\infty}^{\infty} d\tau (\theta^2 - \frac{\omega_0^2\tau^2}{4}) \cos\left[\omega\tau\left\{\frac{\gamma^{-2}+\theta^2}{2} + \frac{\omega_0^2\tau^2}{24}\right\}\right]. \qquad (1)$$

Here P is the power (energy/time), $\omega_0 = c/\rho$ and the θ is an angle

$$\theta^2 = \phi^2 + \psi^2,$$

where ϕ and ψ are horizontal and vertical opening angles, respectively. As is clear from eq.(1), the radiation is symmetric between ϕ and ψ up to the second order.

Usual equation, for example, that used in Refs. [5, 6] is not ready to calculate the horizontal opening angle, because the opening angle and the sweep angle are identified.

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[2] Work supported by Department of Energy contract DE-AC03-76SF00515.

In other words, the horizontal angle is usually integrated away. This is natural if we are concerned with the radiation power seen by an observer located at a position outside the beam. We are here, however, interested in the reaction of the photon emission to the emitting particle. In this case, the usual integrated formula is not useful.

From Eq.(1) and the definition of the Airy function, Eq.(40), we obtain

$$\frac{dP}{d\omega d\phi d\psi} = \frac{\hbar\alpha}{\pi}\omega\left(\frac{\omega}{\omega_0}\right)^{2/3}(2\theta^2+\gamma^{-2})A_i\left(\left(\frac{\omega}{\omega_0}\right)^{2/3}(\theta^2+\gamma^{-2})\right), \qquad (2)$$

where use is made of Eq.(41). Using Eq.(42), it can be cast in a form

$$\frac{dP}{d\omega d\phi d\psi} = \frac{\hbar\alpha}{\pi^2\sqrt{3}\omega_0}\omega^2(2\theta^2+\gamma^{-2})\sqrt{\theta^2+\gamma^{-2}}K_{1/3}\left(\frac{2\omega}{3\omega_0}[\theta^2+\gamma^{-2}]^{3/2}\right). \qquad (3)$$

Energy Spectrum Integrating Eq.(2) over $d\phi d\psi = \pi d\theta^2$, we get the well-know expression

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\hbar\alpha\gamma\omega_0\xi F(\xi), \qquad F(\xi) = \int_\xi^\infty K_{5/3}(\xi')d\xi', \qquad (4)$$

where

$$\xi = \frac{2\omega}{3\gamma^3\omega_0}.$$

Angular Spectrum Integrating Eq.(3) over $d\omega$, and using Eq.(43), we get

$$\frac{dP}{d\phi d\psi} = \frac{\hbar\alpha}{\pi}\omega_0^2\frac{2\theta^2+\gamma^{-2}}{(\theta^2+\gamma^{-2})^4}. \qquad (5)$$

This is a monotonically decreasing function as shown in Fig.1. Note that this distribution has a non-Gaussian tail and decreases only slowly for large θ.

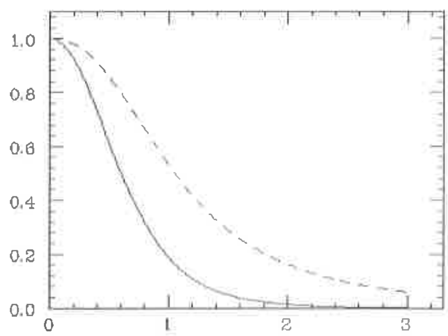

Figure 1: Solid: Function $(2X^2+1)/(X^2+1)^4$, appearing in Eq.(5). Here X stands for $\theta\gamma$. Dashed: Function $(2X^2+1)/(X^2+1)^{5/2}$, appearing in Eq.(10).

Total Power Further integration of Eq.(4) over $d\omega$ or Eq.(5) over $d\phi d\psi$ lead to the total power (energy/time)

$$\int dP \equiv P_{tot} = \frac{2\hbar\alpha c^2\gamma^4}{3\rho^2}. \qquad (6)$$

3 Quantum Formulae

Photon Number We are interested in the energy and angular distribution of each photon. The number of photons (or the number of photon emission) per unit time is given by [1] $dN = dP/(\hbar\omega)$.

$$\frac{dN}{dud\phi d\psi} = \frac{\alpha}{\hbar\pi}\left(\frac{u}{u_0}\right)^{2/3}(2\theta^2+\gamma^{-2})A_i\left(\left(\frac{u}{u_0}\right)^{2/3}(\theta^2+\gamma^{-2})\right), \tag{7}$$

$$= \frac{\alpha}{\hbar\pi^2\sqrt{3}}\frac{u}{u_0}(2\theta^2+\gamma^{-2})\sqrt{\theta^2+\gamma^{-2}}K_{1/3}\left(\frac{2u}{3u_0}[\theta^2+\gamma^{-2}]^{3/2}\right). \tag{8}$$

where $u = \hbar\omega$ and $u_0 = \hbar\omega_0$.

From Eq.(4), just by dividing it by $\hbar\omega$, we have

$$\frac{dN}{du} = \frac{\alpha}{\sqrt{3}\pi\hbar\gamma^2}F\left(\frac{u}{u_c}\right), \tag{9}$$

where

$$u_c = \frac{3\hbar c\gamma^3}{2\rho}.$$

From Eq.(8), by completely parallel way to deriving Eq.(5), we get

$$\frac{dN}{d\phi d\psi} = \frac{\sqrt{3}}{4}\frac{\alpha c}{\pi\rho}\frac{2\theta^2+\gamma^{-2}}{(\theta^2+\gamma^{-2})^{5/2}}. \tag{10}$$

By integrating Eq.(9) over du or Eq.(10) over $d\phi d\psi$, we get the number of the photon emission (per second)

$$N_{tot} = \int dN = \frac{5\sqrt{3}}{6}\frac{\alpha c\gamma}{\rho}. \tag{11}$$

Energy and Angular Distribution of Photons Once a photon is emitted, its energy and angular distribution is given by

$$f(u,\phi,\psi)dud\phi d\psi = \frac{dN}{N_{tot}}dud\phi d\psi. \tag{12}$$

We call the average over f the one-photon average and denote as

$$\langle A\rangle_1 \equiv \int Adud\phi d\psi. \tag{13}$$

The function $n(u)$ used by Sands corresponds to $N_{tot}f$.

As easily seen, for any positive integer n,

$$\langle u^n\phi\rangle_1 = \langle u^n\psi\rangle_1 = \langle u^n\phi\psi\rangle_1 = 0.$$

Interesting quantities are

$$\langle u^2\rangle_1 = \frac{1}{N_{tot}}\frac{55}{24\sqrt{3}}\frac{\alpha\hbar^2c^3}{\rho^3}\gamma^7, \tag{14}$$

$$\langle u^2\phi^2\rangle_1 = \langle u^2\psi^2\rangle_1 = \frac{\langle u^2\theta^2\rangle_1}{2} = \frac{1}{N_{tot}}\frac{13}{24\sqrt{3}}\frac{\alpha\hbar^2c^3}{\rho^3}\gamma^5. \tag{15}$$

The Eq.(14) is well-known. The Eq.(15) was calculated in Ref.[2] for the vertical angle (ψ) in different way. The calculation is easy if we start from Eq.(8) and integrate over u first.

As clear from Eq.(14) and Eq.(15), $\langle u^2\rangle_1$ is roughly γ^2 times as large as $\langle u^2\phi^2\rangle_1 = \langle u^2\psi^2\rangle_1$. Thus the effect of the opening angle becomes smaller for rings with higher energy.

4 Emittances

The diffusion Matrix We use the rectangular coordinate system. The stochastic variable $\boldsymbol{\xi}$ expressing the reaction of a photon emission to the canonical variables is

$$\boldsymbol{\xi}(\boldsymbol{x}) = -\frac{u}{E_0}\left(0, \frac{k_x+\phi}{1+\delta_k}, 0, \frac{k_y+\psi}{1+\delta_k}, 0, 1\right) \tag{16}$$

Since $\langle u\phi\rangle_1 = \langle u\psi\rangle_1 = 0$, the damping matrix[3] is not affected at all by the opening angle. The diffusion matrix has a contribution

$$B(s;\bar{\boldsymbol{x}}) = \frac{\bar{N}(\bar{\boldsymbol{x}},s)\langle u(\bar{\boldsymbol{x}},s)^2\rangle_1}{E_0^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta^2\bar{k}_x^2 & 0 & \Delta^2\bar{k}_x\bar{k}_y & 0 & \Delta\bar{k}_x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta^2\bar{k}_x\bar{k}_y & 0 & \Delta^2\bar{k}_y^2 & 0 & \Delta\bar{k}_y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta\bar{k}_x & 0 & \Delta\bar{k}_y & 0 & 1 \end{pmatrix} + B_0(s;\bar{\boldsymbol{x}}). \tag{17}$$

where $\Delta = 1/(1+\bar{\delta}_k)$ and $\bar{N} = c^{-1}N_{tot}dl/ds$. Here the first term is identical with the B matrix in Ref.[3], while B_0 is the contribution from the opening angle

$$B_0(s;\bar{\boldsymbol{x}}) = \frac{\bar{N}(\bar{\boldsymbol{x}},s)}{E_0^2}\Delta^2 diag(0, \langle u^2\phi^2\rangle_1, 0, \langle u^2\psi^2\rangle_1, 0, 0, 0). \tag{18}$$

Radiation Integrals If we employ the Frenet-Serret base, all of $\bar{k}_x$, $\bar{k}_y$ and $\bar{\delta}_k$ in Eq.(17) vanishes so that we arrive at

$$B = diag(0, B_{22}, 0, B_{22}, 0, B_{66}), \tag{19}$$

$$B_{22} = B_{44} = \frac{\bar{N}(\bar{\boldsymbol{x}},s)}{E_0^2}\langle u^2\phi^2\rangle_1, \qquad B_{66} = \frac{\bar{N}(\bar{\boldsymbol{x}},s)}{E_0^2}\langle u^2\rangle_1. \tag{20}$$

Thus the perturbative "emittances"[3] become

$$(\varepsilon_u, \varepsilon_v, \varepsilon_w) = \frac{1}{2}\left(\frac{I_u}{\alpha_u}, \frac{I_v}{\alpha_v}, \frac{I_w}{\alpha_w}\right). \tag{21}$$

Here α's are the damping rates: these are unaffected by the opening angle and we can use the same expressions as in Ref.[3]. Now we have

$$I_u = \oint ds\left[(|V_{12}|^2 + |V_{14}|^2)B_{22} + |V_{16}|^2 B_{66}\right], \tag{22}$$

$$I_v = \oint ds \left[(|V_{32}|^2 + |V_{34}|^2) B_{22} + |V_{36}|^2 B_{66} \right], \tag{23}$$

$$I_w = \oint ds \left[(|V_{52}|^2 + |V_{54}|^2) B_{22} + |V_{56}|^2 B_{66} \right]. \tag{24}$$

Here V is the diagonalizing matrix. (See Eqs.(96) - (102) in Ref.[3].)

Let us assume that the dispersions are zero in cavities. In this case, the radiation integrals have simpler forms:

$$|V_{12}|^2 = \frac{b^2 \beta_u}{2}, \tag{25}$$

$$|V_{14}|^2 = \frac{r_2^2 + (\alpha_u r_2 - \beta_u r_1)^2}{2\beta_u} \equiv \frac{1}{2}\mathcal{G}_u, \tag{26}$$

$$|V_{16}|^2 = \frac{\eta_u^2 + (\alpha_u \eta_u - \beta_u \eta_u')^2}{2\beta_u} \equiv \frac{1}{2}\mathcal{H}_u, \tag{27}$$

$$|V_{32}|^2 = \frac{r_2^2 + (\alpha_v r_2 + \beta_v r_4)^2}{2\beta_v} \equiv \frac{1}{2}\mathcal{G}_v, \tag{28}$$

$$|V_{34}|^2 = \frac{b^2 \beta_v}{2}, \tag{29}$$

$$|V_{36}|^2 = \frac{\eta_v^2 + (\alpha_v \eta_v - \beta_v \eta_v')^2}{2\beta_v} \equiv \frac{1}{2}\mathcal{H}_v, \tag{30}$$

$$|V_{52}|^2 = \frac{(1+\alpha_w^2)\eta_u^2}{2\beta_w}, \tag{31}$$

$$|V_{54}|^2 = \frac{(1+\alpha_w^2)\eta_v^2}{2\beta_w}, \tag{32}$$

$$|V_{56}|^2 = \frac{\beta_w}{2}. \tag{33}$$

Here $|V_{16}|^2$ and $|V_{36}|^2$ correspond to well-known terms and $|V_{56}|^2$ is shown in Ref.[3].

Perfect Machines Let us assume that the ring is perfect and the dispersions are zero in cavities. In this case, Eqs.(22)-(24) are much simplified: (u, v, w) is identical with (x, y, z) and b is unity.

$$I_u = \oint ds \left[\frac{\beta_x}{2} B_{22} + \frac{\mathcal{H}}{2} B_{66} \right], \tag{34}$$

$$I_v = \oint ds \left[\frac{\beta_y}{2} B_{22} \right], \tag{35}$$

$$I_w = \oint ds \left[\frac{\gamma_z}{2} \eta_x^2 B_{22} + \frac{\beta_z}{2} B_{66} \right], \tag{36}$$

where $\gamma_z = (1+\alpha_z^2)/\beta_z$ and $\mathcal{H} = \mathcal{H}_x$.

Let us evaluate the effects of the opening angle to the emittances by using the smooth approximation. In the smooth approximation, we have $\beta_z = \alpha_m R/\nu_s$, where R is the mean radius, α_m the momentum compaction factor and ν_s the synchrotron tune. For the energy spread $\langle \delta^2 \rangle$, we get

$$\frac{\langle \delta^2 \rangle_O}{\langle \delta^2 \rangle_C} = \frac{\oint ds \gamma_z \eta_x^2 B_{22}}{\oint ds \beta_z B_{66}} \simeq \frac{\nu_s^2}{\gamma^2} \frac{\rho^2}{R^2}. \tag{37}$$

Here the suffix O and C mean "opening angle contribution" and "conventional scheme", respectively and a use is made of $\alpha_m \simeq \eta_x/\rho$ in the last approximate equality. The opening angle contribution is extremely small.

Similarly, the contribution to the horizontal emittance is

$$\frac{\varepsilon_x^O}{\varepsilon_x^C} \simeq \frac{\beta_x}{\mathcal{H}}\frac{B_{22}}{B_{66}} = \frac{26}{55}\frac{\beta_x}{\mathcal{H}}\frac{1}{\gamma^2}. \tag{38}$$

Lastly the vertical emittance ε_y is[2]

$$\varepsilon_y = \frac{13}{55}C_q\frac{\oint ds\frac{\beta_y}{\rho^3}}{\oint ds\frac{1}{\rho^2}} \simeq 0.9\times10^{-13}\times\frac{\beta_y^w}{\rho_w} \quad (\mathrm{m}\cdot\mathrm{rad}). \tag{39}$$

The last approximation applies when the damping time is dominated by the wigglers.

The vertical emittances of some damping rings are evaluated as follows:

	NLC	ATF	JLC
L_b (m)	29	35.8	
L_w (m)	12	9.6	
β_y^b (m)	3	2.6	
β_y^w (m)	6	5.3	4.4
ρ_b (m)	4.6	5.7	
ρ_w (m)	2.5	2.6	2.73
$\varepsilon_y \times10^{-13}$ (m)	1.5	1.2	1.6

The vertical emittances due to the opening angle is less than 2% of the specified values.

5 Discussions

We have extended the diffusion matrix B in the envelope formalism and the radiation integrals by including the opening angle effects. For usual parameters of the storage and damping rings, its effects are tiny. For more future machines with smaller emittances, linear or circular, the envelope formalism including the opening angle effects should play an important role. The same formalism can also be applied to evaluate the emittance growth effects due to other incoherent processes: the scattering by the residual gas, for example.

Acknowledgements

P. Chen and T. Raubenheimmer are acknowledged for useful discussions. He also thanks the members of the Accelerator Theory group of SLAC for their hospitality extended to him.

A Airy and related functions

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du \cos[ux + \frac{u^3}{3}], \tag{40}$$

$$Ai''(x) - xAi(x) = 0 \tag{41}$$

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}(\frac{2}{3}x^{3/2}), \qquad Ai'(x) = -\frac{1}{\pi}\frac{x}{\sqrt{3}}K_{2/3}(\frac{2}{3}x^{3/2}) \tag{42}$$

$$\int_0^{\infty} t^{\mu-1}K_\nu(at)dt = 2^{\mu-2}a^{-\mu}\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right). \tag{43}$$

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