Dear Frank,
Here is the paper.
Cheers, Elias

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LANDAU DAMPING BY NON-LINEARITY

by

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Page

1. Introduction

Transverse coherent instabilities, driven by wake fields, may be stabilised by frequency spread. The case where this spread comes from the longitudinal momentum spread of the beam is straightforward, at least for a coasting beam, because the longitudinal momentum is a constant, which just affects the coefficients in the equations of motion of the transverse oscillations, and hence their frequency. When the frequency spread comes from a nonlinearity of the focusing, the situation is less clear, because the coherent motion is then a small addition to the large incoherent amplitudes that make the frequency spread, and it is inconsistent to assume that it can be treated as a linear superposition, and it is not obvious that the large-amplitude frequency is the appropriate effective resonant frequency for calculating the additional motion due to a small perturbation.

Since the result is quite different from the usual assumption, we try to make it convincing by three distinct attacks; in sections 2, 3, 4 . But the essentials, in a rough concise form, are in Appendix II.

We consider only the case where the non-linear term comes from the external focusing system, not the case where it comes from the beam space-charge.

2. Free oscillations

In this section we study a rather simple situation to show that the usual picture of non-linearity causing frequency to depend on amplitude, then amplitude range making a spread of resonant frequencies, can be misleading. We shall look at the free-ringing frequency of a beam which occupies a region of phase-space with uniform density.

Consider particles that move according to

$$\ddot{x} + \nu_0^2 x + F(x) = 0 \qquad (2.1)$$

here $F(x)$ represents the non-linear part of the restoring force, so we can suppose

$$F(0) = 0, \quad F'(0) = 0$$

and $\frac{2\pi}{\nu_0}$ is the period of small oscillations. For larger oscillations the period will depend upon their amplitude A and one can express the motion of a particle

$$x = A \cos [\nu(A)t + \psi] \quad + \text{higher harmonics} \qquad (2.2)$$

If the non-linearity is fairly small the higher harmonic terms will be so also, and the trajectory of such a particle in the $x, \dot{x}$ plane is roughly an upright ellipse*. All particles on this trajectory go round it once in the period $2\pi/\nu(A)$. In the range of amplitudes from zero to A will be found all periods from $2\pi/\nu_0$ to $2\pi/\nu(A)$.

If one fills the area enclosed by such a trajectory with a uniform density of particles, Liouville's theorem dictates that they all go round with their own amplitudes and periods in such a way that the density distribution is completely independent of time, that is, the amplitude of coherent motion is zero. So let us rather consider a uniform density distribution whose outline is just a little shifted compared with the A trajectory, as illustrated in Fig.1.

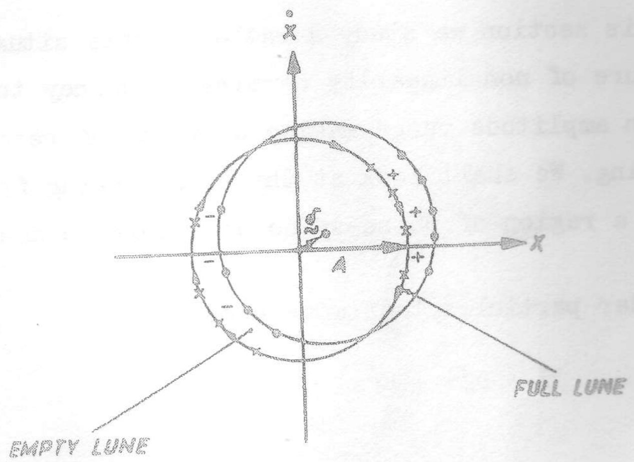

Fig. 1

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* Its semiaxes are A and $A\nu(A)$. They can be made equal by a suitable choice of scale, but not simultaneously for different amplitudes.

The result of the shift $\underset{\sim}{\delta}$ is to make a thin lune-shaped region occupied by particles just outside the A trajectory and an empty lune just inside it; everywhere else the density is as it was in the unshifted case.

Now consider how these lunes move with the passage of time. They move round in the phase plane with the period $2\pi/\nu(A)$. More precisely: the points xxx which constitute their base lie on the A trajectory and therefore go round with period exactly $2\pi/\nu(A)$ for ever, while the points ... lie on nearby values of amplitude and will go round with periods which are near to $2\pi/\nu(A)$ and tend to $2\pi/\nu(A)$ in the limit of small $\underset{\sim}{\delta}$. It is clear that the motion of these lunes determines the motion of the centroid of the whole beam, i.e. the coherent oscillation. Thus we have the somewhat surprising result:

> For a beam having uniform phase-space density for all amplitudes up to some A, the free oscillation period for oscillation of the average x, in the limit that its amplitude is small, is equal to the period of individual particles at amplitude A, and shows no frequency spread.*

Thus the frequency spread of particles that are immersed in a uniform phase-plane density is "invisible", and the average motion discloses only the frequency of the extreme particles. We show in Appendix II how the effective coherent-oscillation frequencies are such as to give this result.

There are some more remarks about our validity "in the limit of small $\underset{\sim}{\delta}$" in Appendix I.

It is worth mentioning that this argument with the uniform density and the lunes applies equally to higher order modes like the quadrupolar (throbbing-beam, beam-envelope, monopole) oscillations.

* By this we mean there is (in the limit of small coherent amplitude) no slow diminution or beating of the coherent amplitude, of the kind that normally results from spectrum width. It remains true that the frequency is impure in the sense of containing higher multiples of $\nu(A)$, coming from the non-linearity.

3. The frequency response

We need to know the response of a large-amplitude particle to a small external perturbation. So we shall solve

$$\ddot{x} + \nu_0^2 x + F(x) = B \exp j\omega t \qquad (3.1)$$

and subtract the appropriate solution of

$$\ddot{x}_0 + \nu_0^2 x_0 + F(x_0) = 0 \qquad (3.2)$$

to obtain

$$x_1 = x - x_0 \qquad (3.3)$$

We assume B small and drop terms higher than first order in it. Choice of the appropriate x_0 avoids any B-independent part of x_1. So finally x_1 will be proportional to B.

The method we use is to consider first the effect of a perturbation in the form of an impulse, replacing (3.1) by

$$\ddot{x} + \nu_0^2 x + F(x) = B\,\delta(t - t_0) \qquad (3.4)$$

Where $\delta(\;)$ is the Dirac delta function.
For the unperturbed (3.2) we write the solution

$$x_0 = A \cos[\nu t + \psi] + \text{hh.} \qquad (3.5)$$

where h.h. represents the higher harmonic terms already mentioned in (2.2). For brevity we are writing an unsubscripted ν for $\nu(A)$. And for t earlier than t_0 (3.4) is the same equation so we give it the same solution:

$$\text{for } t < t_0, \quad x = x_0 \qquad (3.6)$$

For times later than t_o the right hand side of (3.4) is again zero and we take the most general solution near to x_o :

for $t > t_o$,

$$x = (A + \delta A)\cos[\nu(A+\delta A)t + \psi + \delta\psi] + \text{h.h.} \tag{3.6b}$$

The two quantities δA and $\delta\psi$ are constants which are treated as small, and can be found by satisfying (3.4) in the neighbourhood of $t = t_o$.

This requires

$$\begin{aligned} x(t = t_o+) - x(t = t_o -) &= 0 \\ \mathring{x}(t = t_o+) - \mathring{x}(t = t_o -) &= B. \end{aligned} \tag{3.7}$$

Differentiating (3.6) and introducing the abbreviation

$$\frac{A}{\nu}\frac{d\nu}{dA} = K ,$$

(in general K depends on A)

for $t > t_o$ we have

$$\begin{aligned} \mathring{x} = \mathring{x}_o &- \delta A\ \nu\ (1+K)\ \sin(\nu t + \psi) \\ &- \delta A\ \nu^2\ t\ K\ \cos(\nu t + \psi) \\ &- \delta\psi\ A\ \nu\ \cos(\nu t + \psi) \\ &+ \text{h.h.} \end{aligned} \tag{3.8}$$

and (3.7) gives us two equations for δA and $\delta\psi$, with solutions*

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* The quantities δA and $\delta\psi$ depend on t_o , the time of application of the impulse, but they are constants in being independent of the time t .

$$\delta A = B\,\frac{-\sin(\nu t_0+\psi)}{1+K\sin^2(\nu t_0+\psi)} \qquad (3.9)$$

$$\delta\psi = B\,\frac{\cos(\nu t_0+\psi)-K\nu t_0\,\sin(\nu t_0+\psi)}{A\nu\left(1+K\sin^2(\nu t_0+\psi)\right)}\,.$$

Giving for x_1

$$t \leqslant t_0\,,\quad x_1 = 0$$

$$t \geqslant t_0\,,\quad x_1 = \frac{B}{\nu}\,(1-K/2)\sin(t-t_0)\nu + \frac{BK}{2}(t-t_0)\cos(t-t_0)\nu + \text{h.h.} \qquad (3.10)$$

where we have expanded the denominators in (3.9) and retain only as far as the first order in K.

Remember this is for the delta-function perturbation, (3.4); to solve the case with simple-harmonic perturbation (3.1) we must multiply by $\exp j\omega t_0$ and integrate dt_0 from $-\infty$ to t. This is a calculation by linear superposition, justified by the fact that we assume the perturbation B and its effects small enough to work to first approximation in B. The functions integrate easily; evaluation at the upper limit yields

$$x_1 = \left(1-\frac{K}{2}\right)\frac{1}{\nu^2-\omega^2}\,B\exp j\omega t - \frac{K}{2}\,\frac{\nu^2+\omega^2}{(\nu^2-\omega^2)^2}\,B\exp j\omega t + \text{h.h.} \qquad (3.11)$$

The lower limit is more awkward, as there are terms that oscillate infinitely as $t_0 \to -\infty$, but they all converge to zero if we assume that ω has at least a little negative imaginary part [1].

If one puts $K = 0$ (and h.h. = 0) in (3.11), one recovers the familiar linear resonance. The extra factor $(1 - K/2)$ in the first line is possibly only an unimportant correction if K is small, but the second line has a different order of $(\nu - \omega)$ - dependence, and is something new.

4. The uniform-density case

We want to average (3.11) over a uniformly occupied region of phase-space, in order to compare the result with that of section 2. Simplest would be to multiply by πdA^2 *, integrate from zero to A_m, and divide by πA_m^2; but we can easily be a little more precise. Consider the unperturbed oscillation of (3.5):

$$x_0 = A \cos[\nu t + \psi] + \text{h.h.} \qquad (4.1)$$

$$\dot{x}_0 = -A\,\nu \sin[\nu t + \psi] + \text{h.h.}$$

The ellipse that fits the fundamental evidently has area $\pi A^2 \nu$ in the $x, \dot{x}$ plane, and

$$d(\pi A^2 \nu) = 2\pi \left(1 + \frac{K}{2}\right) \nu\, AdA \qquad (4.2)$$

So the response factor averaged over the uniformly filled area, from (3.11) and (4.2), to first order in K, is

* That is by 2π AdA

$$\frac{1}{\pi A_m^2 \nu_m} \int_0^{A_m} \left(\frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu(\nu^2 + \omega^2)}{(\nu^2 - \omega^2)^2} \right) 2\pi \, A dA \tag{4.3}$$

where ν_m stands for $\nu(A_m)$.

This can be integrated. Consider

$$\frac{d}{dA} \frac{\nu A^2}{\nu^2 - \omega^2}$$

$$= \frac{2A\nu}{\nu^2 - \omega^2} - A^2 \frac{\nu^2 + \omega^2}{(\nu^2 - \omega^2)^2} \cdot \frac{d\nu}{dA}$$

$$= \left(\frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu(\nu^2 + \omega^2)}{(\nu^2 - \omega^2)^2} \right) 2A \tag{4.4}$$

So (4.3) is equal to

$$\frac{1}{A_m^2 \nu_m} \left[\frac{\nu A^2}{\nu^2 - \omega^2} \right]_0^{A_m}$$

$$= \frac{1}{\nu_m^2 - \omega^2} \tag{4.5}$$

provided ω is not equal to $\pm \nu_0$ or $\pm \nu_m$. If ω is real and lies in the range of ν we get zero denominators in (4.3), and can deform the path of integration or give ω a little imaginary part to avoid them [1]. This does not change the result (4.5); the pole has zero residue. Thus we have again the result that the uniformly filled region behaves as though all the particles had a resonant frequency corresponding to the period at the greatest amplitude.

Such a system has no Landau damping. However, real beams do not have a uniform phase-space density with a sharp cut-off, so we must extend the theory to more general density distributions before seeing its practical implications.

5. Non-uniform density

Let the phase-space density be given by $\rho(A)$. The number of particles present is then, from (4.2)

$$N = \int_0^\infty \rho(A) \quad d(\pi A^2 \nu)$$

$$= \int_0^\infty \left(1 + \frac{K}{2}\right) \nu \cdot \rho(A) \quad 2\pi A dA \tag{5.1}$$

and the response factor averaged over the particles is the appropriate modification of (4.3) :

$$\frac{1}{N} \int_0^\infty \left\{ \frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu(\nu^2 + \omega^2)}{(\nu^2 - \omega^2)^2} \right\} \rho(A) \; 2\pi A dA \tag{5.2}$$

It is interesting to integrate this by parts. Call

$$A \left\{ \quad \right\} = v'$$

and $\quad \rho(A) \;\; 2\pi \quad = \quad u$

then (5.2) is equal to

$$\frac{1}{N}\left[\ u\ \ v\ \right]_0^\infty - \frac{1}{N}\int_0^\infty u'\ v\ dA \tag{5.3}$$

The first term vanishes at both limits provided $\omega \neq \nu_0$ and $\rho(\infty)$ is small enough. Using (4.4) we are left with

$$-\frac{\pi}{N}\int_0^\infty \rho'(A)\ \frac{\nu\ A^2}{\nu^2-\omega^2}\ dA \tag{5.4}$$

Notice that $d\rho/dA$, not ρ, is the weighting function inside this integral : as in section 4, regions of uniform density contribute nothing. As a check, one can recover (4.5) by putting $\rho' = -\delta(A-A_m)$ into (5.4). If ρ' is non-zero at $\nu = \omega$ the usual small deformation of the path of integration is necessary and (5.4) yields an imaginary, Landau-damping, part.

Our ρ' in the dispersion integral (5.4) agrees with Laslett et al.[3], who found it by using the Vlasov-equation method; but others who use a more direct approach usually fail to find it.

6. Acknowledgements

I have to thank P.Morton for reading a preliminary version of the text. He pointed out that the spread of frequencies from non-linearity may be effective in giving Landau damping in the case of two- or three-dimensional oscillations, even if the phase-space density is rather uniform. I also had some useful comments from D.Möhl and E.Merle.

Appendix I

Small perturbations

No apology is needed for calculating the case of small perturbations (small $\underset{\sim}{\delta}$ on page 3, small B, δA, $\delta\psi$, in section 3), for we are mainly interested in criteria for stability of stationary initial distributions, and for this the behaviour in the limit of small perturbations is primordial. One leaves until later the behaviour of unstable oscillations as they grow into the region of non-small amplitude, and the possibility of finite oscillations growing in a system where small ones do not. Some care is however necessary with the time variable. For instance, when comparing x with x_0 on page 5 we effectively made use of

$$\cos\left[\nu(A + \delta A)t + \psi\right]$$

$$= \cos\left[\nu(A)t + \psi\right] - \delta\nu t \sin\left[\nu(A)t + \psi\right] \qquad (I.1)$$

It is true that $\frac{d}{dA}\cos[\nu(A)t + \psi]$ is $-\frac{d\nu}{dA}t\sin[\nu(A)t + \psi]$, so (I.1) is correct in the limit of small δA. But clearly it is in fact

$$|\delta\nu.t| \ll 1 \qquad (I.2)$$

that is the condition for (I.1) to be a good approximation.

So we may say:

- For any $|t|$ one can choose $\delta\nu$ small enough to make the approximation good.
- But, for any $|\delta\nu|$ greater than zero the approximation will fail at sufficiently large $|t|$.

This sort of behaviour is typical of approximate solutions to differential equations.

Exactly parallel remarks apply to the behaviour of the lunes on page 2 :

- For any $|t|$ we can choose $\underset{\sim}{\delta}$ small enough that they go round $t\,\nu(A)/2\pi$ times, with negligible phase error and deformation.

- But, for any $|\underset{\sim}{\delta}|$ greater than zero they will be grossly deformed and phase-shifted at sufficiently large $|t|$.

For any specific case of instability, with known growth rate and some information on the minimum coherent amplitude needed for practical recognition, it is possible to check numerically whether our small perturbation approximation is good. If $\delta\nu$ is changing slowly, our condition for validity (by a slight generalisation of (I.2)) is

$$\int \delta\nu . dt \ll 1 \qquad (I.3)$$

and if $\delta\nu$ has been growing exponentially for at least a few time-constants, this is

$$\delta\nu . \tau \ll 1 \qquad (I.4)$$

where τ is the timeconstant.

Similarly if the theory predicts stability with a certain damping rate, one can calculate up to what level external stimulation or noise can be considered to be small.

Another difficulty is that (3.11) gives a particle amplitude that tends to infinity as ν tends to ω, so it is not obvious that one can make B small enough for our approximation to be good for <u>all</u> particles. However, the behaviour of particles whose ν-dependence on amplitude brings them close or on to such a resonance is known from non-linear machine theory [2]. To the usual approximation there is in the rotating phase-plane a stable and an unstable fixed point at the resonant amplitude, with topology like Fig. 2.

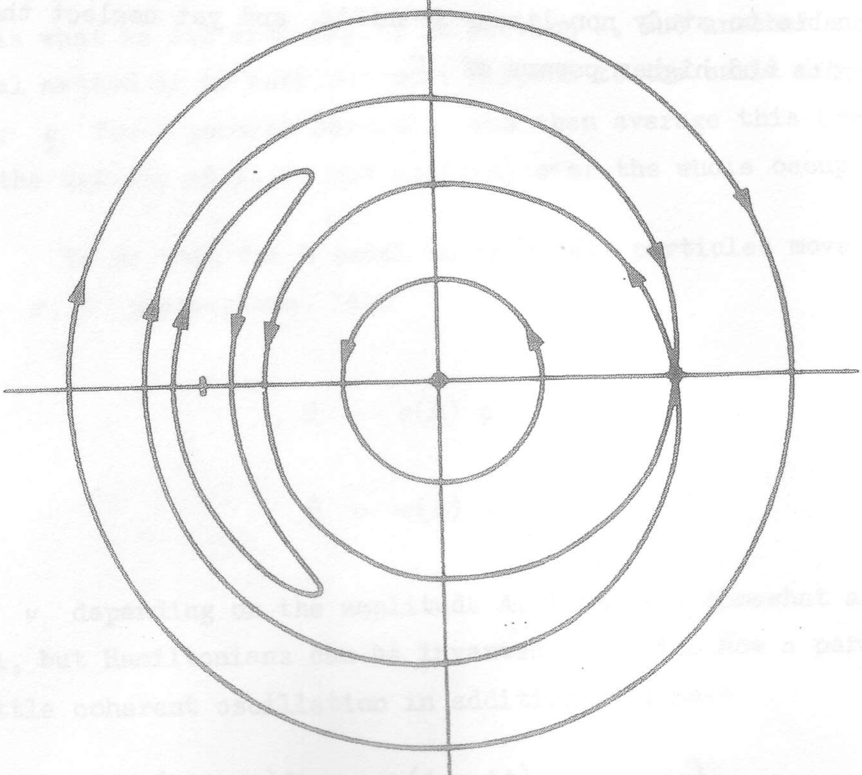

Fig. 2

No particles go to unlimited amplitude, even at large t, and such a pattern can be immersed in a region of uniform phase-space density without causing anything special in the behaviour of the average x.

Finally a remark on the systematic disregard of the higher harmonic terms and the fact that we have worked only to the first order in the coefficient K. In a machine with a high Q-value, or in the medium and high field range of a machine with Q of only 5 or so, it is quite common for the Q spread and all the real and complex Q shifts to be very small compared with Q itself. Then the machine is practically linear in the sense that K is small and so are the non-linear harmonic terms in the oscillations. Yet at the same time the non-linear Q-spread may be about as big as the other real or complex shifts associated with a possible instability, so is by far not negligible in that connexion. In such cases it is perfectly reasonable to study non-linear Q-shifts and yet neglect the higher harmonics and higher powers of K.

Appendix II

Average frequency

The behaviour of the centroid of a region occupied by particles can be inferred, when the density is uniform, from the boundaries alone. This is what we did with fig.1, in section 2. But another and more general method is to work out what happens to the small displacement vector $\underset{\sim}{\delta}$ for a general particle, and then average this (weighted with the density if it is not uniform) over the whole occupied area.

We do this for a model in which all particles move in circles in an x, p phase-plane, like

$$\begin{aligned} \dot{x} &= \nu(A)\, p \\ \dot{p} &= -\nu(A)\, x \end{aligned} \qquad \text{(II.1)}$$

with ν depending on the amplitude A. This is a somewhat artificial model, but Hamiltonians can be invented to do it. Now a particle with a little coherent oscillation in addition will have

$$\begin{aligned} (x + \delta x)^{\cdot} &= \nu(A + \delta A) \,.\, (p + \delta p) \\ (p + \delta p)^{\cdot} &= -\nu(A + \delta A) \,.\, (x + \delta x) \end{aligned} \qquad \text{(II.2)}$$

Subtracting and working to first order in δ-quantities we get

$$\begin{aligned} \delta\dot{x} &= \nu(A)\delta p + p\delta\nu \\ \delta\dot{p} &= -\nu(A)\delta x - x\delta\nu \end{aligned} \qquad \text{(II.3)}$$

where

$$\begin{aligned} \delta\nu &= \nu(A+\delta A) - \nu(A) \\ &= \frac{d\nu}{dA}\,\delta A \\ &= \frac{\nu K}{A}\,\delta A \end{aligned} \qquad \text{(II.4)}$$

from the definition of K.

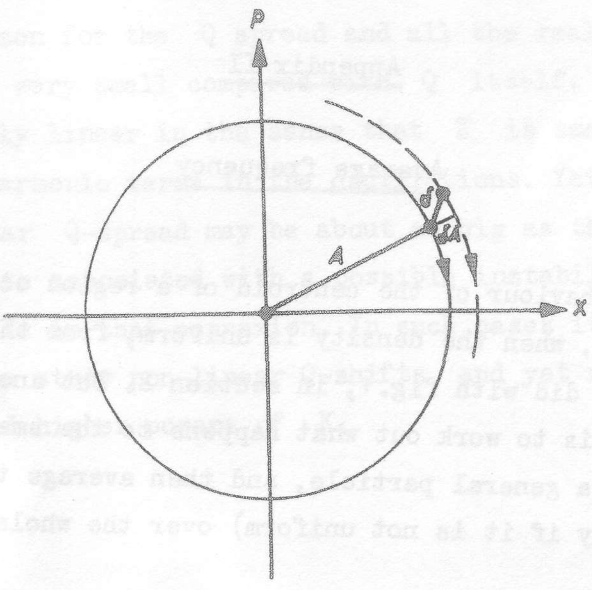

Fig. 3

From the geometry (Fig. 3) by resolving the vector δx, δp along the direction x, p,

$$\delta A = (x\delta x + p\delta p)/A \tag{II.5}$$

and so (II.3) becomes

$$\begin{aligned} \mathring{\delta x} &= \nu\delta p + K\nu(px\delta x + p^2\delta p)/A^2 \\ \mathring{\delta p} &= -\nu\delta x - K\nu(x^2\delta x + xp\delta p)/A^2 \end{aligned} \tag{II.6}$$

We now take a set of particles all with the same δx, δp, and A, but with x and p corresponding to a uniform distribution in phase round the A circle. Averaging over these particles gives

$$\begin{aligned} px &\rightarrow 0 \\ x^2 &\rightarrow A^2/2 \\ p^2 &\rightarrow A^2/2 \end{aligned} \qquad (II.7)$$

and (II. 6), so averaged, gives

$$\begin{aligned} \mathring{\delta x} &= \nu(1 + K/2)\delta p \\ \mathring{\delta p} &= -\nu(1 + K/2)\delta x \end{aligned} \qquad (II.8)$$

So $\nu(1 + K/2)$ takes the place of ν as the effective frequency for the small coherent motion of the amplitude-A particles.

Now consider a uniform density of particles from amplitude zero up to A_m. It is not completely clear what is the interpretation of an average frequency, but we can anyway calculate it :

$$\left[\nu\,(1 + K/2)\right]_{Av} = \frac{1}{A_m^2} \int_0^{A_m} \nu\,(1 + K/2)\; 2A dA$$

Now we use

$$\frac{d}{dA}(\nu A^2) = 2A\nu + A^2 \frac{d\nu}{dA}$$

$$= 2A\nu\,(1 + K/2)$$

to evaluate the integral and find

$$\left[\nu(1 + K/2)\right]_{Av} = \frac{1}{A_m^2}\left[\nu A^2\right]_0^{A_m} = \nu(A_m)$$

Thus the extra term $\nu K/2$ is just enough to bring the average value up to the extreme ν, $\nu(A_m)$.

References

1) See, for example, Weber, Linear transient analysis, Vol. II, ch.1. Or Hereward, The elementary theory,......, CERN 65-20, ch.III.

2) See, for example, Schoch, Theory of linear and non-linear ...... CERN 57-21, Fig.1, facing page 26. I have not found a paper that includes a non-linear phase-plane diagram for the vicinity of a resonance of order n = 1.

3) Laslett, Neil and Sessler, Rev. Sci. Instr. 36 436 (1965).

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