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NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH
Section A

Vertical synchrobetatron resonances due to beam–beam interaction with horizontal crossing

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Due to a nonlinearity of the beam–beam kick, the crossing-angle collisions of bunches in the horizontal plane can cause the coupling between the synchrotron and vertical betatron oscillations of particles. These resonances can be of the same strength as the coupling beam–beam resonances between vertical and horizontal betatron oscillations. A common influence of the crossing in the horizontal plane and of the phase-averaging effect on the vertical synchrobetatron resonances is briefly discussed.

1. Introduction

There are many arguments why an angle crossing is a desirable operational option for colliders in which a high luminosity must be obtained using multi-bunch beams. The performance of schemes with the crossing in the vertical plane is limited by resonances between vertical and synchrotron oscillations of particles [1]. For this reason, the designs of the future B-factories mainly focus on the schemes with the crab- or conventional crossing in the horizontal plane (see, for instance, refs. [2,3]). Since the crab-crossing is a very new technique, the schemes with a conventional crossing in the horizontal plane are presently considered as a number one for practical applications. As an additional advantage, it is expected that the crossing in the horizontal plane will excite only the coupling resonances between the horizontal betatron and synchrotron oscillations [4].

In this paper we calculate the strengths of the synchrobetatron beam–beam resonances with the excitation of vertical betatron oscillations for the crossing in the horizontal plane. These resonances occur due to a nonlinear dependence of the beam–beam deflecting force on the transverse offsets of a particle. If ϕ is a half crossing angle, a_s amplitude of the synchrotron and a_x amplitude of the horizontal betatron oscillations, then the strengths of these synchrobetatron resonances coincide with the strengths of the beam–beam betatron coupling resonances, calculated for $a_x = \phi a_s$. In a collider with flat bunches, like B-factory, relevant instability of the vertical betatron oscillations can limit the acceptable value of the crossing angle. In colliders with round bunches, these resonances can result in additional limitations to those, which were found in ref. [4], on the position of the working point in the tune space.

We assume the weak–strong beam approximation, zero dispersion function in the interaction region and neglect chromatic effects.

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2. Strengths of resonances for short bunches

For short colliding bunches ($\sigma_s \ll \beta_z^*$), where σ_s the length of the strong bunch and β_z^* the value of the vertical β-function at the interaction point (IP), we assume that in the interaction region the oscillations of a particle near the closed orbit are given by the following equations:

$$\begin{aligned}
&z = \sqrt{J_z\beta_z(\theta)}\cos(\psi_z), \quad x = \sqrt{J_x\beta_x(\theta)}\cos(\psi_x),\\
&s = R_0\theta = ct + R_0\varphi, \quad \varphi = \varphi_s\cos\psi_s, \quad \Delta p = -\frac{p_0\nu_s}{\eta}\varphi_s\sin\psi_s,\\
&R_0(p_z/p) = dz/d\theta = z', \quad R_0(p_x/p) = x', \quad \Delta p = p - p_0,\\
&\psi_z' = \nu_z, \quad \psi_x' = \nu_x, \quad \psi_s' = \nu_s,\\
&I_{z,x} = \frac{pJ_{z,x}}{2}, \quad I_s = \frac{pR_0\nu_s\varphi_s^2}{2|\eta|} = \frac{pJ_s}{2}.
\end{aligned} \tag{1}$$

Here, $\Pi = 2\pi R_0$ is the perimeter of the closed orbit, $\eta = (1/\gamma^2) - \alpha$, α momentum compaction factor, and $p_0 \simeq \gamma Mc$ momentum of the synchronous particle. Equations of motion of the test particle from the weak beam are generated by the Hamiltonian:

$$H(J,\psi,\theta) = \nu_x J_x + \nu_z J_z + \nu_s J_s - V_{bb}(J,\psi,\theta), \tag{2}$$

where the term V_{bb} describes the perturbations due to the beam–beam interaction. For a Gaussian distribution $\rho(\boldsymbol{r}_\perp)$ over transverse coordinates in the strong bunch:

$$\rho(\boldsymbol{r}_\perp) = \frac{N}{2\pi\sigma_x\sigma_z}\exp\left(-\frac{(x+\phi s)^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right), \tag{3}$$

and ultra-relativistic $\gamma \gg 1$ electron and positron bunches, colliding at the angle 2ϕ, we can write V_{bb} in the following form [#1]

$$V_{bb} = \frac{4Ne^2R_0}{pc}\lambda(s+ct)\int\frac{d^2k}{\pi k^2}\exp\left\{i[k_x(x+2\phi s) + k_z z] - \frac{k_x^2\sigma_x^2 + k_z^2\sigma_z^2}{2}\right\} \tag{4}$$

Here, $\lambda(s)$ is a linear density in the strong bunch. Eq. (4) describes the perturbation of the weak bunch as a sequence of the periodic and very short (during $\Delta t \simeq \sigma_s/c$) kicks. The strengths of resonances due to this perturbation is estimated by the values of the amplitudes of the following Fourier expansion

$$V_{bb} = \sum_{m,n} V_{m,n}\exp(i[m_x\psi_x + m_z\psi_z + m_s\psi_s - n\theta]). \tag{5}$$

Here, m denotes the combinations $\{m_x, m_z, m_s\}$. Since

$$J_\alpha' = -\delta_\alpha J_\alpha + \frac{\partial V_{bb}}{\partial\psi_\alpha}, \quad \psi_\alpha' = \nu_\alpha - \frac{\partial V_{bb}}{\partial J_\alpha}, \quad \alpha = x, z, s, \tag{6}$$

where δ_α are the (dimensionless) decrements due to, say, synchrotron radiation damping, the amplitudes ($\sqrt{J_\alpha}$) and phases (ψ_α) of oscillations get systematic variations when the tunes (ν_x, ν_z and ν_s) approach the resonant values

$$m_x\nu_x + m_z\nu_z + m_s\nu_s = n. \tag{7}$$

Generally, the rates of these systematic variations are determined by $|V_{m,n}|$.

For short bunches ($\sigma_s \ll \beta_z^*$) we may neglect in eq. (4) the modulation of β_x and β_z along the interaction region. Then, the azimuthal harmonics of V_{bb} are determined by the integral

[#1] This expression can be easily obtained taking into account that indeed $\phi \ll 1$. Then, a recalculation of V_{bb} from the reference system, where the strong bunch has no transverse velocity, results in eq. (4).

$$v_n = \int_0^{\Pi} \frac{ds}{\Pi} \lambda(2s - R_0\varphi) \exp(-ik_x 2\phi s + ins/R_0)$$

$$\simeq \exp(-ik_x\phi R_0\varphi + in\varphi/2) \int_{-\infty}^{\infty} \frac{du}{2\Pi} \lambda(u) \exp(-ik_x\phi u + inu/R_0)\,. \tag{8}$$

Typically, the resonant harmonic numbers (n) are not very high. If, for example, the amplitude

$$(a_s)_{\max} = R_0(\varphi_s)_{\max} \gg \sigma_s$$

determines the longitudinal aperture of the ring due to the beam–beam resonances, then one has $|n|(\varphi_s)_{\max} \ll 1$. For such harmonic numbers we may substitute in eq. (8) $\exp(-in\varphi/2) \simeq 1$ and $\exp(inu/R_0) \simeq 1$. This gives the following expression for v_n

$$v_n \simeq \frac{\exp(-ik_x\phi R_0\varphi)}{2\Pi} \lambda(k_x\phi)\,. \tag{9}$$

If we take as $\lambda(s)$ a Gaussian distribution, then $\lambda(k_x\phi) = \exp(-k_x^2\phi^2\sigma_s^2/2)$ and we obtain

$$v_n \simeq \frac{1}{2\Pi} \exp\left(-ik_x\phi R_0\varphi - \frac{(k_x\phi\sigma_s)^2}{2}\right). \tag{10}$$

Using the definition of $V_{m,n}$, we can write

$$V_{m,n} = \frac{4Ne^2R_0}{pc} \int_0^{2\pi} \frac{d^3\psi}{(2\pi)^3} \exp(-i[m_x\psi_x + m_z\psi_z + m_s\psi_s])$$

$$\times \int \frac{d^2k}{\pi k^2} v_n(k_x) \exp\left\{i(k_x x + k_z z) - \frac{k_x^2\sigma_x^2 + k_z^2\sigma_z^2}{2}\right\}.$$

Substituting here $v_n(k_x)$ from eq. (10), the expansion

$$\exp(ika\cos\psi) = \sum_{m=-\infty}^{\infty} i^m J_m(ka)\, e^{im\psi} \tag{11}$$

and defining $\Sigma_x^2 = \sigma_x^2 + \phi^2\sigma_s^2$, we find (a constant phase factor, which is not important here, is omitted)

$$V_{m,n} \simeq \frac{Ne^2}{\pi pc} \int \frac{d^2k}{\pi k^2} J_{m_x}(k_x a_x) J_{m_z}(k_z a_z) J_{m_s}(k_x\phi a_s) \exp\left\{-\frac{k_x^2\Sigma_x^2 + k_z^2\sigma_z^2}{2}\right\}. \tag{12}$$

Eq. (12) predicts two sets of synchrobetatron resonances for a scheme with the crossing in the horizontal plane. First, these are the resonances with an excitation of the horizontal betatron oscillations ($m_z = 0$). They are described by the amplitudes (to simplify expressions we take $a_z = 0$)

$$V_{m,n} \simeq \frac{Ne^2}{\pi pc} \int \frac{d^2k}{\pi k^2} J_{m_x}(k_x a_x) J_{m_s}(k_x\phi a_s) \exp\left\{-\frac{k_x^2\Sigma_x^2 + k_z^2\sigma_z^2}{2}\right\}. \tag{13}$$

It is important that except for the resonances with $m_x = 2l$, and hence, $m_s = 2q$, where $q = 0, \pm1, \pm2, \ldots$, eq. (13) describes the resonances of the odd order ($m_x = 2l + 1$ and $m_s = 2q + 1$). So that if, for example, $m_x = 1$, the possible values of m_s are $m_s = \pm1, \pm3, \ldots$. This result coincides with results of simulations in ref. [4]. For very flat bunches ($\sigma_z \ll \Sigma_x$) we can put in eq. (13) $\sigma_z = 0$. Then, the integration over k_z results in

$$V_{m,n} \simeq \frac{2Ne^2}{\pi pc} \int_0^{\infty} \frac{dk}{k} J_{m_x}\left(k\frac{a_x}{\Sigma_x}\right) J_{m_s}\left(k\frac{\phi a_s}{\Sigma_x}\right) \exp(-k^2/2). \tag{14}$$

For small amplitudes ($J_x \ll \epsilon_x^{\rm eff}$, where $\epsilon_x^{\rm eff} = \Sigma_x^2/(\beta_x^*)$ and for $m_x = 1$ we can substitute in eq. (14) $J_1(ka_x/\Sigma_x) \simeq ka_x/(2\Sigma_x)$. This yields

$$V_{1,m_s,n} \simeq 2\xi_x\epsilon_x^{\rm eff}\sqrt{\frac{J_x}{\epsilon_x^{\rm eff}}}\mathcal{I}_{m_s/2}\left(\frac{\phi a_s}{2\Sigma_x}\right), \quad J_x \ll \epsilon_x^{\rm eff}. \tag{15}$$

Here,

$$\xi_x = \frac{Ne^2}{2\pi pc\epsilon_x^{\rm eff}}$$

is the beam–beam parameter for horizontal oscillations, $\mathcal{I}_m(x) = {\rm e}^{-x^2}I_m(x^2)$, and $I_m(x)$ is the Bessel function of the imaginary argument [5].

Another set of synchrobetatron resonances is described by the amplitudes with $m_x = 0$ and even m_s (now we take $a_x = 0$)

$$V_{m,n} \simeq \frac{Ne^2}{\pi pc}\int\frac{{\rm d}^2k}{\pi k^2}J_{m_z}(k_z a_z)J_{m_s}(k_x\phi a_s)\exp\left\{-\frac{k_x^2\Sigma_x^2 + k_z^2\sigma_z^2}{2}\right\}. \tag{16}$$

These are resonances with the excitation of the vertical betatron oscillations. Except for the difference between σ_x and Σ_x, the amplitudes $V_{m,n}$ coincide with similar amplitudes for the coupling beam–beam resonances $m_x\nu_x + m_z\nu_z = n$, calculated for the amplitude $a_x = \phi a_s$. For this reason, these resonances have equal strengths for equivalent amplitudes.

Simple expression for the amplitudes $V_{m,n}$ can be obtained in the region $\sigma_z \ll a_z \ll \Sigma_x$ (see appendix):

$$V_{m,n} \simeq 2\xi_z\epsilon_z\sqrt{\frac{J_z}{\epsilon_z}}\frac{2^{3/2}}{\sqrt{\pi}(m_z^2-1)}\mathcal{I}_{m_s/2}\left(\frac{\phi a_s}{2\Sigma_x}\right), \tag{17}$$

Note, that due to very flat geometry of the strong bunch, eq. (17) holds for both integer ($m_z = 2$) and higher order resonances. This means that in the region $a_x \sim a_z \ll \Sigma_x$ the perturbations due to vertical synchrobetatron resonances will dominate. Direct comparison of eqs. (15) and (17) shows that in equivalent conditions the rates of the variations of $J_x/\epsilon_x^{\rm eff}$ (due to an integer resonance $m_x = 1$) and of J_z/ϵ_z differ not very much, if $\phi a_s/\Sigma_x \geq 1$ (see also fig. 1). This figure shows that factors $\mathcal{I}_{m_s}$ almost coincide for horizontal and vertical resonances in the region $\phi a_s/\Sigma_x \geq 5$. As can be seen, in the region $\phi a_s/\Sigma_x \leq 1$ vertical synchrobetatron sidebands are well suppressed as compared to the strength of the corresponding betatron resonance ($I_0(x) \simeq 1$, if $x \to 0$). A suppression of the strengths ($V_{m,n}$) at small amplitudes of the synchrotron oscillations is typical for synchrobetatron resonances.

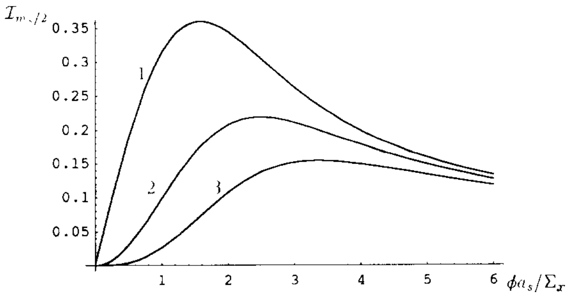

Fig. 1. Dependencies of the strengths of the synchrobetatron resonances on the amplitude of synchrotron oscillations; (1). $m_x = 1$, $m_s = 1$; (2). $m_z = 2$, $m_s = 2$; (3). $m_x = 1$, $m_s = 3$.

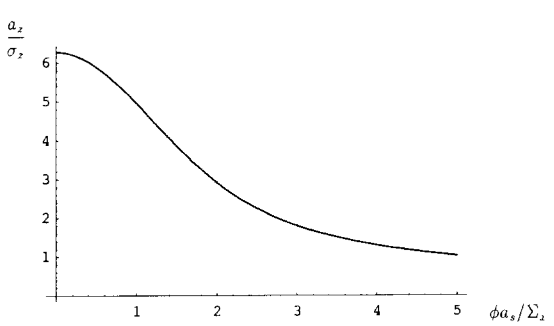

Fig. 2. Resonance line (a_z vs a_s) for vertical oscillations; $\xi_z/|\Delta| = 5$.

3. Vertical phase-space near isolated resonances

The fact that the strengths of the vertical resonances are determined by a simple expression in eq. (17) enables the calculation of the phase trajectories of vertical oscillations near isolated resonances and a direct evaluation of their widths in both the amplitude- and tune-space. The motion near isolated betatron resonance in slow variables (J_z and $\chi = \psi_z - n\theta/m$) is described by the reduced Hamiltonian

$$H = \Delta x + 2\xi\,[F + V\cos(m_z\chi)]\sqrt{x}, \quad x = J_z/\epsilon_z, \tag{18}$$

where

$$F = \sqrt{\frac{\pi}{2}}\mathcal{I}_0\left(\frac{\phi a_s}{2\Sigma_x}\right), \quad V = \frac{2^{3/2}}{\sqrt{\pi}(m_z^2-1)}\mathcal{I}_0\left(\frac{\phi a_s}{2\Sigma_x}\right). \tag{19}$$

It can have fixed points:

$$\frac{\partial H}{\partial \chi} = 0, \quad \frac{\partial H}{\partial J_z} = 0. \tag{20}$$

The first equation yields $\sin(m_z\chi) = 0$, which specifies the so-called $H_\pm$-Hamiltonians:

$$H_\pm = \Delta x + 2\xi(F \pm V)\sqrt{x}. \tag{21}$$

These are simple parabolic curves which have maxima, if $\Delta \leq 0$. In this case, the second equation from eqs. (20) results in

$$y_{st}^\pm = \frac{\xi_z}{|\Delta|}(F \pm V). \tag{22}$$

As can be seen from eq. (18), if $\Delta \leq 0$, betatron oscillations have a bucket in the slow phase-space. The center of this bucket is placed at y_{st}^+, while its width (δy_b) is determined by the equation

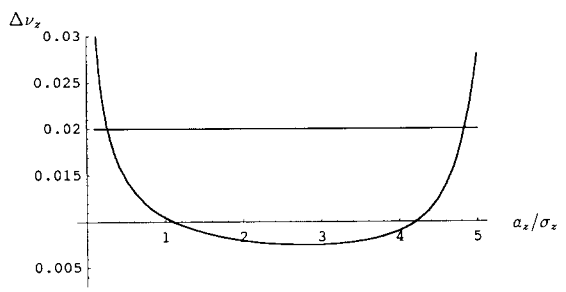

Fig. 3. Schematic dependence of the beam–beam tune shift on amplitude of betatron oscillations near a difference-type synchrobetatron resonance; $m_z = m_s$, $(a_s)_{in} = 0$.

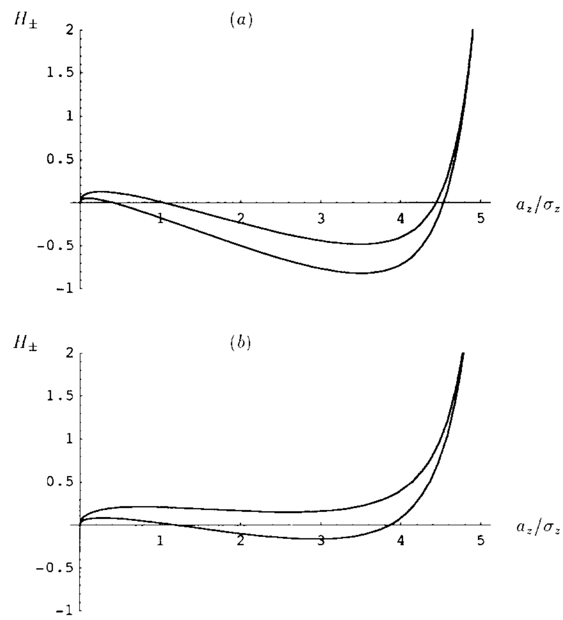

Fig. 4. Schematic dependencies of (a) H_+ and (b) H_- on amplitude of betatron oscillations near a difference-type synchrobetatron resonance; $m_z = m_s$, $(a_s)_{in} = 0$, $\xi_z = 0.05$, (a) $\Delta = -0.025$, (b) $\Delta = -0.015$.

$$H_-(y_{st}^-) = H_+(y),$$

which gives

$$\delta y_b = \frac{4\xi_z}{|\Delta|}\sqrt{FV}. \tag{23}$$

The (dimensionless) frequency of small phase oscillations in the bucket ($\delta\nu_b$) can be easily calculated using eqs. (18) and (20). The result is:

$$\delta\nu_b = |m_z\Delta|\sqrt{\frac{V}{F+V}}. \tag{24}$$

The synchrotron radiation damping does not destroy the resonance, if $\delta\nu_b \gg \delta_z$.

Due to the dependence of the beam–beam tune shift on the amplitude of synchrotron oscillations

$$\Delta\nu_z(J_z, a_s) = \frac{\xi_z F(a_s)}{\sqrt{x}}, \tag{25}$$

the resonance condition ($\Delta\nu_z(J_z, a_s) = -\Delta$) defines a resonant line in the space (a_z, a_s) (see fig. 2). As can be seen from eq. (18), near a pure betatron resonance ($m_s = 0$), synchrotron oscillations are not affected by the perturbation. This means that after a longitudinal deflection of a particle, its J_s decreases as

$$J_s(\theta) = (J_s)_{in}\exp(-\delta_s\theta).$$

Provided that $\delta\nu_b \gg \delta_z$, the particle moves in the plane (a_z, a_s) along the resonant line, which increases the amplitude of its betatron oscillations. Such a blow-up due to the so-called phase-convection effect is specific for two-dimensional resonances [10].

The description of the oscillations near an isolated synchrobetatron resonance ($m_z\nu_z + m_s\nu_s = n$) can be reduced to the study of an equivalent one-dimensional problem using an additional integral of motion

$$\frac{J_z}{m_z} - \frac{J_s}{m_s} = C. \tag{26}$$

In the case of the sum-type resonance ($m_z m_s > 0$) eq. (26) and the resonance condition give one resonant value of $(J_z)_{st}$ and, therefore, only one bucket around this point.

In the case of the difference-type resonance ($m_z m_s < 0$) as can be seen from fig. 3, the resonance condition ($\Delta\nu_z(J_z, a_s) = -\Delta$) generally yields two stationary amplitudes. This determines two buckets in the betatron phase-space. One around the slow phase

$$\chi = \psi_z - \frac{m_s}{m_z}\psi_s - \frac{n}{m_s}\theta = 0,$$

and another one around $\chi = \pi$. As seen from figs. 3 and 4, the distance between these buckets decreases and buckets become wider, when $|\Delta|$ decreases. Fig. 3 also shows that given value of C determines a threshold value of Δ, when the resonance conditions cannot be held anymore. This describes a well known fact that on the difference-type resonance the amplitudes of the coupled oscillations vary within a limited range. In this sense, the accumulation of particles in buckets near the difference-type synchrobetatron resonances does not limit the dynamic aperture of the ring, but results in the limitation on the peak value of the luminosity of a collider.

4. Strengths of resonances for long bunches

Let us now briefly discuss the influence of the length of the strong bunch on the strengths of the vertical synchrobetatron resonances. In the case $\sigma_s \geq \beta_z^*$ and for flat bunches ($\sigma_x \gg \sigma_z$, $\beta_x^* \gg \beta_z^*$), in the calculations of harmonics $V_{m,n}$ we must take into account in eqs. (2) the modulation of the phase of vertical betatron oscillations due to

$$\frac{d\psi_z}{ds} = \frac{d\chi_z}{ds} + \frac{\nu_z}{R_0} = \frac{1}{\beta_z(s)}. \tag{27}$$

For a resonance $m_z\nu_z + m_s\nu_s = n$, after the calculation the harmonics of V_{bb} over the phases of betatron oscillations using eq. (11), we can rewrite eq. (12) in the form (for the sake of simplicity we take $a_x = 0$)

$$V_{m,n} \simeq \frac{2Ne^2}{\pi pc} \int \frac{d^2k}{\pi k^2} \int_0^{2\pi} \frac{d\psi_s}{2\pi} e^{-im_s\psi_s} \int_{-\infty}^{\infty} ds\, \lambda(2s - R_0\varphi)$$

$$\times J_{m_z}(k_z\sqrt{J_z\beta_z(s)}) \exp\left\{-\frac{k_x^2\sigma_x^2 + k_z^2\sigma_z^2}{2} - ik_x 2\phi s + im_z\psi_z(s)\right\}. \tag{28}$$

In the region $\sigma_z \ll a_z \ll \sigma_x$ due to a fast convergence of the integral over k_x, we can neglect in eq. (28) the variation of $1/k^2$ with k_x. Remaining integration over k_x yields

$$\int_{-\infty}^{\infty} dk_x \exp\left\{-\frac{k_x^2\sigma_x^2}{2} - ik_x 2\phi s\right\} = \frac{\sqrt{2\pi}}{\sigma_x} \exp\left\{-\frac{2\phi^2 s^2}{\sigma_x^2}\right\}. \tag{29}$$

Substituting this expression in eq. (28), we obtain

$$V_{m,n} = Y_m^f V_{m,n}^{(0)}. \tag{30}$$

Here,

$$V_{m,n}^{(0)} = 2\xi_z\epsilon_z\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{dv}{v^2} J_{m_z}\left(v\sqrt{\frac{J_z}{\epsilon_z}}\right) e^{-v^2/2}, \quad m_z = 2l, \tag{31}$$

is the strength of the resonance calculated for the synchronous particle due to head-on collisions with a short bunch, and

$$Y_m^f = 2\int_0^{2\pi} \frac{d\psi_s}{2\pi} e^{-im_s\psi_s} \int_{-\infty}^{\infty} ds\, \lambda(2s - R_0\varphi) \exp\left\{-\frac{2\phi^2 s^2}{\sigma_x^2} + im_z\psi_z(s)\right\} \sqrt{\frac{\beta_z(s)}{\beta_z^*}} \tag{32}$$

is the so-called resonance suppression factor [6–8]. For a Gaussian linear density in the strong bunch we can rewrite eq. (32) in the form

$$Y_m^f = \sqrt{2/\pi} \int_0^{2\pi} \frac{d\psi_s}{2\pi} e^{-im_s\psi_s}$$

$$\times \int_{-\infty}^{\infty} du \exp\left(-2\left[u - \frac{a_s\cos\psi_s}{2\sigma_s}\right]^2 - \frac{2\phi^2\sigma_s^2u^2}{\sigma_x^2} + im_z\psi_z(\zeta u)\right)\sqrt{1 + \zeta^2u^2}, \tag{33}$$

$$\zeta = \sigma_s/\beta_z^*.$$

This expression only by a factor $\exp(-2\phi^2\sigma_s^2u^2/\sigma_x^2)$ in the integrand differs from similar expressions, calculated for head-on collisions of long bunches. For this reason, eqs. (31) and (32) describe the excitation of the vertical synchrobetatron resonances with both even and odd m_s. The last possibility ($m_s = 2q + 1$) is a specific feature of the collisions of long bunches. Since in the region $\phi\sigma_s \ll \sigma_x$ this factor exhibits a very wide dependence on u, the behaviour of the strengths of resonances ($\propto Y_m^f$) will differ from that, calculated for head-on collisions, only by small corrections ($\propto (\phi\sigma_s/\sigma_x)^2$) (figs. 5 and 6, see also in ref. [9]). In particular, for small crossing angles a strong suppression of both betatron and synchrobetatron resonances can be predicted for core particles, if $\sigma_s \simeq \beta_z^*$; for tail particles ($a_s \gg$

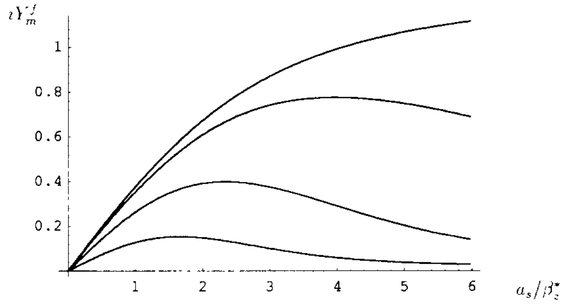

Fig. 5. Dependence of $Y^{\rm f}_m$ on amplitude of synchrotron oscillations; $\sigma_s = \beta_z^*$, $\sigma_x/\sigma_s = 0.01$, $m_z = 2$, $m_s = 1$, from top to bottom: $\phi = 0$, 0.002, 0.005, 0.01.

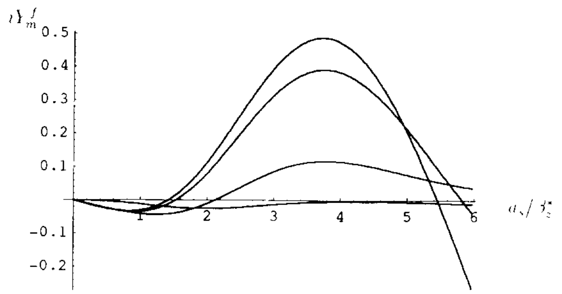

Fig. 6. Dependence of $Y^{\rm f}_m$ on amplitude of synchrotron oscillations; $\sigma_s = \beta_z^*$, $\sigma_x/\sigma_s = 0.01$, $m_z = 8$, $m_s = 1$, from top to bottom: $\phi = 0$, 0.002, 0.005, 0.01.

β_z^*) the strengths the synchrobetatron resonances can reach (or, even exceed) the nominal values [8]. Figs. 5 and 6 show an additional suppression of the vertical synchrobetatron resonances in the region $a_s \leq \sigma_s$ with an increase in ϕ.

Dependencies of the vertical beam–beam tune shift ($\Delta\nu_z$) on the amplitude of synchrotron oscillations and on the crossing angle are described by $Y^{\rm f}_0$. Fig. 7 shows that with the increase in the crossing angle the dependence of $\Delta\nu_z$ on a_s varies from increasing (for head-on collisions) to decreasing (at $\phi \geq 0.005$). As seen, in the region $\phi \simeq 0.0025$ (and for aspect ratio 0.01) this dependence becomes very weak, which can eliminate for long bunches the blow-up of the beam due to the phase-convection mechanism.

In the region $\phi\sigma_s \gg \sigma_x$ the value of the integral over u in eq. (34) is mainly determined by the region $|u| \leq \sigma_x/(\phi\sigma_s) \ll 1$, where we can take

$$\exp\left(-2\left[u - \frac{a_s \cos\psi_s}{2\sigma_s}\right]^2\right) \simeq \exp\left(-\frac{a_s^2\cos^2\psi_s}{2\sigma_s^2}\right).$$

This results in the following expression

$$Y^{\rm f}_m = \frac{\sigma_x}{\phi\sigma_s}\mathcal{I}_{m_s}(a_s/\sigma_s)\,(Y^{\rm f}_m)_0, \tag{34}$$

where the factor $(Y^{\rm f}_m)_0$ coincides with a resonance suppressing factor of the synchronous particle ($a_s = 0$), calculated for the bunch length $\sigma_s' = \sigma_x/\phi$:

$$(Y^{\rm f}_m)_0 = \sqrt{2/\pi}\int\limits_{-\infty}^{\infty} du \exp[-2u^2 + im_z\psi_z(\zeta_\phi u)]\sqrt{1+\zeta_\phi^2 u^2}, \quad \zeta_\phi = \frac{\sigma_x}{\phi\beta_z^*}. \tag{35}$$

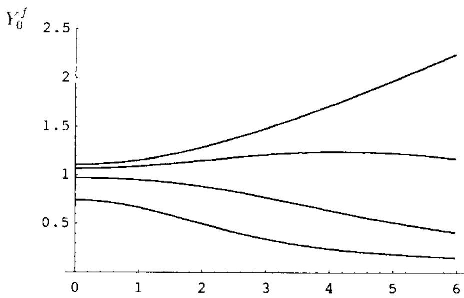

Fig. 7. Dependence of $Y^{\rm f}_m$ on amplitude of synchrotron oscillations; $\sigma_s = \beta_z^*$, $\sigma_x/\sigma_s = 0.01$, from top to bottom: $\phi = 0$, 0.0025, 0.005, 0.01.

If $\sigma_s \simeq \beta_z^*$, then, $\zeta_\phi \ll 1$ and

$$(Y_m^f)_0 \simeq 1, \quad Y_m^f \simeq \frac{\sigma_x}{\phi\sigma_s}\mathcal{I}_{m_s}(a_s/\sigma_s)\,. \tag{36}$$

This case corresponds to a strong reduction of the luminosity due to both the increase of the effective spot sizes of bunches and dynamical limitations.

For all (except, maybe, integer and parametric) resonances the values of $(Y_m^f)_0$ become very small [8] in the region $\beta_z^* \simeq \sigma_x/\phi \ll \sigma_s$, which corresponds to very long bunches, or to a micro-β lattice. In this region the value of the luminosity will be determined by limitations due to horizontal synchrobetatron resonances.

5. Discussion

A conventional crossing of colliding bunches at the angle in the horizontal plane is accompanied by the excitation of both horizontal and vertical synchrobetatron resonances and, generally, increases the dimension of the beam–beam resonances. These synchrobetatron resonances are especially important for the tail particles of the bunch ($\phi a_s > \Sigma_x$). In the case of the round colliding bunches, a new family of synchrobetatron resonances results in additional limitations on the position of the working point of the ring.

In the case of flat bunches ($\sigma_x \gg \sigma_z$) the condition $\phi a_s/\Sigma_x \leq 1$ is not so severe like similar condition for schemes with the vertical crossing. However, an accumulation of particles in the buckets of vertical oscillations near these resonances can saturate the luminosity of the collider, if the perturbation due to horizontal synchrobetatron resonances is small. If the perturbation due to horizontal synchrobetatron resonances is strong, such an accumulation will result in the decrease of the luminosity. If $|\Delta| \ll \xi_z$, vertical synchrobetatron resonances produce buckets well outside the core of the beam ($a_z \gg \sigma_z$). This can cause a deviation of the vertical tails of bunches from Gaussian tails. Phase oscillations in the bucket can produce new families of beam–beam resonances.

The dependence of the beam–beam tune shifts on the amplitude of synchrotron oscillations near betatron resonances results in the increase of amplitudes of betatron oscillations due to the phase-convection effect. Due to quantum fluctuations of the synchrotron radiation this can cause additional blow-up of the transverse sizes of bunches near betatron beam–beam resonances. This effect is specific for short bunches ($\sigma_s \ll \beta_z^*$), or, if $\sigma_s \simeq \beta_z^*$, for large crossing angles.

For long bunches ($\sigma_s \simeq \beta_z^*$), crossing in the horizontal plane at small collision angle ($\phi \ll \sigma_x/\sigma_s$) has a little effect on the strengths of beam–beam resonances. The features of the beam–beam instability, in this case, are mainly determined by the phase-averaging effect and by the modulation of β-functions along the interaction region.

Crossing at the large collision angle ($\phi \gg \sigma_x/\sigma_s$) results in the reduction of the luminosity due to both geometric and dynamic effects, if $\sigma_s \sim \beta_z^*$. A reduction of the strengths of vertical resonances is possible for micro-β lattices when $\beta_z^* \simeq \sigma_x/\phi \ll \sigma_s$. In this case the luminosity will be limited by the horizontal synchrobetatron resonances.

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Appendix A. Calculation of $V_{m,n}$ for vertical oscillations

Let us estimate the amplitudes $V_{m,n}$ in eq. (16) in the region

$$\sigma_z \ll a_z \ll \Sigma_x . \tag{A.1}$$

Due to a fast convergence of the integral over k_z, we can put in eq. (16) $\sigma_z = 0$. This yields

$$V_{m,n} \simeq \frac{Ne^2}{\pi pc} \int_{-\infty}^{\infty} \mathrm{d}k_x j_m(u) J_{m_s}(k_x \phi a_s/\Sigma_x) \exp(-k_x^2/2),$$

$$j_m(u) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z J_{m_z}(k_z u)}{\pi(k_x^2 + k_z^2)}, \quad u = a_z/\Sigma_x . \tag{A.2}$$

Using the formula [5]

$$J_m(x) = \int_0^{\pi} \frac{d\psi}{\pi} \cos(x \sin\psi + m\psi), \quad m = 2l,$$

we transform $j_m(u)$ into the following expression

$$j_m(u) = \frac{1}{|k_x|} \int_0^{\pi} \frac{\mathrm{d}\psi}{\pi} \cos m_z\psi \exp(-|k_x| u \sin\psi) . \tag{A.3}$$

In the region (A.1) we can substitute in eq. (A.3)

$$\exp(-k_x u \sin\psi) = 1 - k_x u \sin\psi + \mathcal{O}(u^2) . \tag{A.4}$$

This results in

$$j_m(u) \simeq -u \int_0^{\pi} \frac{\mathrm{d}\psi}{\pi} \cos m_z\psi \sin\psi = \frac{2u}{\pi} \frac{1}{m_z^2 - 1} . \tag{A.5}$$

Substituting this expression in eq. (A.2), we obtain

$$V_{m,n} \simeq \frac{Ne^2}{\pi pc} \frac{2a_z}{\Sigma_x \pi} \frac{1}{m_z^2 - 1} \int_{-\infty}^{\infty} \mathrm{d}k_x J_{m_s}(k_x \phi a_s/\Sigma_x) \exp(-k_x^2/2) . \tag{A.6}$$

The integral

$$\mathcal{I}_{m_s/2}\left(\frac{\phi a_s}{2\Sigma_x}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k_x J_{m_s}(k_x \phi a_s/\Sigma_x) \exp(-k_x^2/2), \quad m_s = 2l',$$

is expressed in terms of the Bessel function of the imaginary argument [5]:

$$\mathcal{I}_m(x) = \mathrm{e}^{-x^2} I_m(x^2) . \tag{A.7}$$

Defining also

$$\xi_z = \frac{Ne^2 \beta_z^*}{2\pi pc \sigma_z \Sigma_x}, \tag{A.8}$$

we rewrite eq. (A.6) in the following form

$$V_{m,n} \simeq 2\xi_z \sqrt{J_z \epsilon_z} \frac{2^{3/2}}{\sqrt{\pi}(m_z^2 - 1)} \mathcal{I}_{m_s/2}\left(\frac{\phi a_s}{2\Sigma_x}\right) . \tag{A.9}$$

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