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THE ABEL-TYPE INTEGRAL TRANSFORMATION WITH THE KERNEL $(t^2 - x^2)^{-\frac{1}{2}}$ AND ITS APPLICATION TO DENSITY DISTRIBUTIONS OF PARTICLE BEAMS

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ABSTRACT

The inversion and some properties of the Abel-type integral transformation with the Kernel $(t^2 - x^2)^{-\frac{1}{2}}$ are demonstrated. The application of this transformation to the different phase-space projections of a coasting beam, whose particles perform two-dimensional pseudo-harmonic oscillations is shown and some frequently-used distributions and their projections tabulated. In a similar way, the correlation between surface densities and projected densities for rotational symmetric systems is given, with examples for the most important cases. Some useful transformation pairs are tabulated in the Appendix.

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1. INTRODUCTION

In order to increase the information output of results obtained with a large spectrum of beam observation devices, we have to invert an integral equation of the type

$$g(x) = \int_x^R \frac{f(t)\,dt}{\sqrt{t^2 - x^2}}\,, \qquad (0 \le x \le R) \tag{1}$$

where g(x) is a given function and f(t) is to be determined. (Usually we have $2R \geq \Phi_{max}$, where Φ_{max} denotes the largest diameter of the beam.) To demonstrate how frequently the above equation applies, a few examples will be enumerated.

Let us consider first the one-dimensional density distribution $\rho(x)$ produced by a set of harmonic oscillators, oscillating around x = 0 with amplitudes given by an amplitude distribution f(a). The contribution of each oscillator to the density at x is $(a^2 - x^2)^{-\frac{1}{2}}/\pi$. Thus, the density $\rho(x)$ produced by all of them results as

$$\rho(x) = \frac{1}{\pi} \int_x^R \frac{f(a)\,da}{\sqrt{a^2 - x^2}}\,, \tag{2}$$

where R denotes the upper limit of the amplitudes. Equation (2) is currently used for the determination of the betatron amplitude distribution of the particles in a synchrotron[1-4], where often $\rho(x)$ is observed instead of f(a). We then have either to invert Eq. (2), or to approximate $\rho(x)$ with functions whose transforms are known.

A second general class of measurements requiring the inversion of Eq. (1) are all those measurements on rotationally symmetric beams, where the surface density P(r) of any physical property can be measured only indirectly by its projection p(x) onto a one-dimensional space. This projection p(x) equals the integral of P(r) over a straight line with the distance x from the centre (see the figure below):

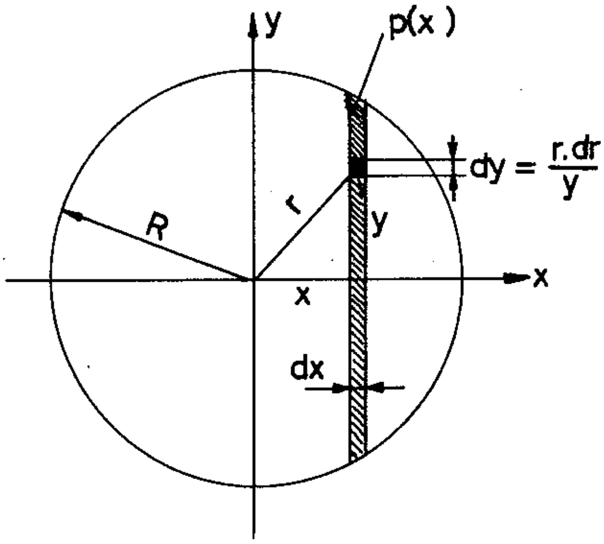
$$p(x) = \int_{-R}^{+R} P(r)\,dy = 2\int_x^{+R} \frac{P(r)r\,dr}{\sqrt{r^2 - x^2}}\,. \tag{3}$$

$$\int_{-R}^{+R} P(r)\,dy = 2\int_0^R P(r)\,dy = 2\int_0^R P(r)\frac{dy}{dr}\,dr = 2\int_x^R \frac{P(r)\,r\,dr}{\sqrt{r^2-x^2}}$$

$$\frac{dy}{dr} = \frac{d\sqrt{r^2-x^2}}{dr} = \frac{r}{\sqrt{r^2-x^2}} \qquad r = \sqrt{x^2+y^2}$$

$y = 0 \Rightarrow r = x$

$y = R \Rightarrow R$



Identifying 2rP(r) with f(t), and p(x) with g(x), we obtain again the Abel-type integral equation (1). It is worth pointing out that, since P(r) represents the surface density dQ/dF of a physical quantity Q, the radial density n(r)

$$n(r) = \frac{dQ}{dr} = 2\pi r\ P(r)$$

is connected with p(x), following Eq. (3), by

$$p(x) = \frac{1}{\pi} \int_x^R \frac{n(r)\,dr}{\sqrt{r^2 - x^2}}$$

in just the same way as is the amplitude distribution f(a) with the density distribution $\rho(x)$ in the first example. Since many electrical and most of the optical measurements performed on beams and plasma jets belong to this class, the Abel-transform (1) becomes quite important, and its inversion by analogue devices has been achieved[5].

2. INVERSION AND PROPERTIES OF THE INTEGRAL TRANSFORMATIONS $g(x) = \int_x^R f(t)\,dt/\sqrt{t^2 - x^2}$ and $h(x) = \int_0^x f(t)\,dt/\sqrt{x^2 - t^2}$

Let us introduce the abbreviation:

$$g(x) = \int_x^R \frac{f(t)\,dt}{\sqrt{t^2 - x^2}} =: K(x,t)\ f(t) \quad , \qquad (4)$$

where K(x,t) stands for the linear integral operator in the above equation which is of the homogeneous Volterra type of the first kind. In a similar way we can define the dual operator J(x,t):

$$h(x) = \int_0^x \frac{f(t)\, dt}{\sqrt{x^2 - t^2}} =: J(x,t)\, f(t) \ . \tag{5}$$

It might be noted that g(x) depends only on the values f(t) over the range $x \le t \le R$, whereas the function h(x) depends on f(t) over the range $0 \le t \le x$.

The difficulty lies in the fact that the non-Hermitian Kernels are not analytic over the whole interval (0,R) due to the two branch points at $t = \pm x$. Furthermore, it will be shown that the existence of the inverse is only ensured for such functions f(t) leading to transforms g(x) and h(x), which are at least differentiable once. Unfortunately, we cannot perform the proof of the differentiability of g(x) and h(x) by the differentiation of the integrals (4) and (5) with the usual rule, since there are singularities at the variable limits of the integration domains. In order to avoid a long discussion of the restrictions on f(t), which ensure that the above equations have unique inverses, we shall make the following assumptions:

i) f(t) is integrable over the range (0,R)

ii) the functions g(x) and h(x) exist and are at least once differentiable on (0,R).

The inversion of the transformation (4) becomes quite easy with the development of the Erdélyi-Kober operators on fractional integration[6-9]. To avoid the inversion with these not very familar operators (see Appendix I), we make use of the method given by Srivastav[10], which is applicable to the inversion of equations of the general forms:

$$\int_x^b \frac{f(t)\, dt}{[\phi(t) - \phi(x)]^\alpha} = g(x) \tag{6}$$

$$\int_a^x \frac{f(t)\, dt}{[\phi(x) - \phi(t)]^{1-\alpha}} = h(x) \ , \tag{7}$$

where f(t) is integrable over (a,b), $0 < \alpha < 1$, and $\phi(t)$ is a strictly monotonically increasing function in (a,b).

To invert Eq. (6) we introduce the integral

$$G(x) = \int_x^b \frac{\phi'(\xi)\, g(\xi)\, d\xi}{[\phi(\xi) - \phi(x)]^{1-\alpha}}$$

and substitute for $g(\xi)$ the expression (6). Interchanging the order of integration and replacing the variable ξ by $\phi(\xi)$ we obtain

$$G(x) = \int_x^b f(t)\, dt \int_{\phi(x)}^{\phi(t)} [\phi - \phi(x)]^{\alpha-1} [\phi(t) - \phi]^{-\alpha}\, d\phi ,$$

where the integral with respect to ϕ can be identified with the beta function $B(\alpha, 1-\alpha) = \Gamma(\alpha) \cdot \Gamma(1-\alpha)$. We have, therefore,

$$\int_x^b f(t)\, dt = \frac{1}{\Gamma(\alpha)\, \Gamma(1-\alpha)}\, G(x)$$

and finally

$$f(x) = -\frac{1}{\Gamma(\alpha)\, \Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{\phi'(\xi)\, g(\xi)\, d\xi}{[\phi(\xi) - \phi(x)]^{1-\alpha}} \tag{8}$$

as the inverse of Eq. (6). In this way we get with Eq. (7)

$$\int_a^x \frac{\phi'(\xi)\, h(\xi)\, d\xi}{[\phi(x) - \phi(\xi)]^{1-\alpha}} = \int_a^x f(t)\, dt \int_{\phi(t)}^{\phi(x)} \frac{d\phi}{[\phi - \phi(t)]^{\alpha} [\phi(x) - \phi]^{1-\alpha}}$$

and hence

$$f(x) = \frac{1}{\Gamma(\alpha)\, \Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{\phi'(\xi)\, h(\xi)\, d\xi}{[\phi(x) - \phi(\xi)]^{1-\alpha}} \tag{9}$$

for the inverse of Eq. (7).

Setting $\phi(\xi) = \xi^2$, $\alpha = \frac{1}{2}$, $a = 0$, $b = R$, Eqs. (8) and (9) become

$$f(x) = -\frac{2}{\pi} \frac{d}{dx} \int_x^R \frac{\xi\, g(\xi)\, d\xi}{\sqrt{\xi^2 - x^2}} = K^{-1}(x, \xi)\, g(\xi) \tag{10}$$

$$f(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{\xi\, h(\xi)\, d\xi}{\sqrt{x^2 - \xi^2}} = J^{-1}(x, \xi)\, h(\xi) , \tag{11}$$

whereas Eqs. (6) and (7) reduce to (4) and (5), respectively. We have therefore found the inverse operators $K^{-1}(t,x)$ and $J^{-1}(t,x)$ to $K(x,t)$ and $J(x,t)$ defined in Eqs. (4) and (5).

Since owing to the fact that the differentiation of the integral cannot be performed by the usual rule, the expressions (10) and (11) are not very comfortable for practical calculation, we shall derive some further properties of the integral operator K(x,t). At first, we see from Eqs. (10) and (11):

$$K^{-1}(x,t) = -\frac{2}{\pi}\frac{d}{dx}K(x,t)t \tag{12}$$

$$J^{-1}(x,t) = \frac{2}{\pi}\frac{d}{dx}J(x,t)t \;. \tag{13}$$

Let us now prove the following statement:

$$K^{-1}(x,t) = -\frac{2}{\pi}x\,K(x,t)\frac{d}{dt}\;, \tag{14}$$

which allows many simplifications in practical computations of the inverses. Starting from

$$g(x) = K(x,t)\frac{d}{dt}f(t)\;, \tag{15}$$

we apply the inverse operator $K^{-1}(t,x)$ and obtain, comparing the differentials on both sides

$$f(t) = -\frac{2}{\pi}K(t,x)x\,g(x)\;.$$

A second application of $K^{-1}(x,t)$ produces

$$g(x) = \frac{1}{x}\frac{d}{dx}K(x,t)t\,f(t)\;.$$

Comparing this with Eq. (15) yields the identity

$$K(x,t)\frac{d}{dt} = \frac{1}{x}\frac{d}{dx}K(x,t)t \tag{16}$$

from which we can immediately derive relation (14). In the same way we obtain

$$J^{-1}(x,t) = \frac{2}{\pi}x\,J(x,t)\frac{d}{dt}\;. \tag{17}$$

It can further be demonstrated that

$$-\frac{2}{\pi}\frac{d}{dx}K(x,t)t\,K(t,x) = I \tag{18}$$

$$\frac{2}{\pi}\frac{d}{dx}J(x,t)t\,J(t,x) = I\;, \tag{19}$$

where I denotes the identity operator. A further useful relation has been derived by Erdélyi[6] and is known as "fractional integration by parts":

$$\int_0^R x\,g(x)\,K(x,t)\,f(t)\,dx = \int_0^R x\,f(x)\,J(x,t)\,g(t)\,dx\;. \tag{20}$$

Finally we have the normalization theorem:

$$\int_0^R K(x,t)\, f(t)\, dx = \frac{\pi}{2}\int_0^R f(t)\, dt \ , \qquad (21)$$

which can be proved by setting g(x) = K(x,t) f(t) and entering f(t) = -(2/π)(d/dt)K(t,x)xg(x) into the right-hand side of Eq. (21). Thus we get

$$\int_0^R f(t)\, dt = -\frac{2}{\pi}\int_{t=0}^{t=R} dF(t) = \frac{2}{\pi}\left[F(0) - F(R)\right]$$

with

$$F(t) = \int_t^R \frac{g(x)x\, dx}{\sqrt{x^2 - t^2}} \ .$$

Then Eq. (21) follows, since F(R) = 0 and

$$F(0) = \int_0^R g(x)\, dx = \int_0^R K(x,t)\, f(t)\, dx \ .$$

Equation (21) can also be written in the form

$$\int_0^R dx \int_x^R \frac{f(t)\, dt}{\sqrt{t^2 - x^2}} = \frac{\pi}{2} \ .$$

The generalization of the normalization theorem for n independent variables, i.e.

$$\int_0^R dx_1 \dots \int_0^R dx_n\, K(x_1, t_1) \dots K(x_n, t_n)\, f(t_1, \dots, t_n)$$
$$= \left(\frac{\pi}{2}\right)^n \int_0^R dt_1 \dots \int_0^R dt_n\, f(t_1, \dots, t_n) \qquad (22)$$

can be demonstrated by induction. For n = 1 we have proved the above statement with Eq. (21). Now let us assume that

$$\int_0^R dx_1 \dots \int_0^R dx_{n-1}\, K(x_1, t_1) \dots K(x_{n-1}, t_{n-1})\, f(t_1, \dots, t_{n-1}, t_n)$$
$$= \left(\frac{\pi}{2}\right)^{n-1} \int_0^R dt_1 \dots \int_0^R dt_{n-1}\, f(t_1, \dots, t_{n-1}, t_n) = F(t_n)$$

is true. Integration of $F(t_n)$ over t_n, and application of Eq. (21) yields

$$\int_0^R dx_n\, K(x_n, t_n)\, F(t_n) = \frac{\pi}{2}\int_0^R dt_n\, F(t_n)$$

and therefore Eq. (22) is also true.

3. CHARGE DENSITY AND CURRENT DENSITY IN BEAMS WITH PSEUDO-HARMONIC OSCILLATIONS

Assume a coasting beam of particles, which perform pseudo-harmonic oscillations around the s-axis described by

$$\begin{aligned} x &= \lambda\sqrt{\beta_x}\cos\left[\psi_x(s)+\alpha\right] \\ z &= \mu\sqrt{\beta_z}\cos\left[\psi_z(s)+\beta\right] \\ s &= v(t+\tau_0) = v\tau \end{aligned} \tag{23}$$

with

$$\psi_k(s) = \int_{s_0}^{s} \frac{ds}{\beta_k(s)} \qquad k = x, z \ . \tag{24}$$

To derive a general expression of the current and charge density, we can make use of a formalism derived elsewhere[11] in connection with the self-consistent calculation of space-charge flows, and which will be summarized briefly for our purpose.

Let us consider τ_0, α, β, λ, μ, ν as the six constants of the motion for each individual particle moving in a conservative system, and let us assume that in the integrated equations of motion [e.g. Eq. (22)] the quantities t and τ_0 appear only in the form $t + \tau_0$. We can then regard $\tau = t + \tau_0$, α, β, λ, μ, ν as generalized coordinates of a six-dimensional space Λ in which the particles move with the constant velocity $\dot{\tau} = 1$, parallel to the τ-axis. Therefore, the density W in this space can depend on t only in the following form:

$$W = f(\tau - t, \alpha, \beta, \lambda, \mu, \nu) \ . \tag{25}$$

For a stationary flow, W depends neither on τ nor on t. Thus W represents only the distribution function of the particles over the five parameters α, β, λ, μ, ν, i.e. over each possible trajectory in the real space.

As the density of the Λ-space, W has to be normalized over Λ:

$$qN = \underset{\Lambda\text{-space}}{\iiiint\!\!\iint} W \, dt \, d\alpha \, d\beta \, d\lambda \, d\mu \, d\nu \ , \tag{26}$$

where qN represents the total charge being considered. For a stationary flow, where W is constant with respect to τ, N becomes infinite and we should rather perform the normalization to the finite quantity $q(dN/d\tau)$ suppressing the integration over the τ-range.

If we take τ, α, β as a subset of these new coordinates, we can regard Eqs. (23) for every set of chosen constants (λ,μ,ν) as a transformation of the real space into a generalized three-dimensional subspace {τ,α,β} of Λ:

$$\vec{x} = \vec{x}_{\lambda\mu\nu}(\tau,\alpha,\beta) \qquad (\text{with } \vec{x} = \{x, z, s\}) \ . \tag{27}$$

Under the assumption that the Jacobian $\sqrt{g} = \partial(x,z,s)/\partial(\tau,\alpha,\beta)$ of the transformation (27) does not vanish, the real space-charge density ρ can be obtained by

$$\rho = M \iiint \frac{W}{\sqrt{g}} \, d\lambda \, d\mu \, d\nu \tag{28}$$

and the current density j in the s-direction by

$$j = M \iiint \frac{\partial s}{\partial t} \frac{W}{\sqrt{g}} \, d\lambda \, d\mu \, d\nu \ , \tag{29}$$

if we replace the coordinates τ, α, β by x, z, s, λ, μ, ν, solving Eq. (27) for τ, α, β before evaluating the integrals (28), (29). M denotes the multiplicity of the transformation (27) between the subspace $\{\tau,\alpha,\beta\} \subset \Lambda$ and the real space R^3, i.e. the number of points in the subspace which belong to the same point in the R^3.

Now let us perform this for the motion given by Eqs. (23). For the Jacobian of the "transformation" (23) we get

$$\sqrt{g} = \frac{\partial(x, z, s)}{\partial(\tau, \alpha, \beta)} = \lambda\mu\nu\sqrt{\beta_x\beta_z} \ \sin(\psi_x + \alpha) \ \sin(\psi_z + \beta) \ . \tag{30}$$

Solving Eq. (23) for τ, α, β we obtain

$$\begin{aligned} \tau &= s/\nu & -\infty < \tau < \infty \\ \alpha &= \arccos\left(\frac{x}{\lambda\sqrt{\beta_x}} - \psi_x\right) & -\pi < \alpha \le +\pi \\ \beta &= \arccos\left(\frac{z}{\mu\sqrt{\beta_z}} - \psi_z\right) & -\pi < \beta \le +\pi \ , \end{aligned} \tag{31}$$

which, introduced in Eq. (30), yields

$$\sqrt{g} = \nu\sqrt{\lambda^2\beta_x - x^2} \ \sqrt{\mu^2\beta_z - z^2} \ . \tag{32}$$

The multiplicity M equals 4 (within the range $-\pi < \alpha, \beta \le +\pi$) and the functions ρ, j can be calculated by

$$\rho(\vec{x}, t) = 4 \iiint \frac{W}{\nu\sqrt{\lambda^2\beta_x - x^2}\,\sqrt{\mu^2\beta_z - z^2}}\, d\lambda\, d\mu\, d\nu \tag{33}$$

$$\vec{j}(\vec{x}, t) = 4 \iiint \frac{W}{\sqrt{\lambda^2\beta_x - x^2}\,\sqrt{\mu^2\beta_z - z^2}}\, d\lambda\, d\mu\, d\nu \tag{34}$$

if we specify the distribution function W.

Let us suppose that the oscillations (23) in the beam are incoherent. Then the distribution W will not depend on the phases α and β. We make the further assumption that the distribution F of the velocity in the s-direction is the same over the whole beam at t = 0, therefore only a function of ν. This implies a distribution function W of the form

$$W = Q_0(\nu\tau - \nu t)\, \nu F(\nu)\, \frac{1}{4\pi^2}\, G(\lambda, \mu)\ , \tag{35}$$

where $G(\lambda,\mu)$ denotes the distribution function of the normalized transverse amplitudes λ, μ. $Q_0 = Q_0(s - \nu t)$ gives us the initial charge distribution along the s-axis at t = 0. If the distributions $F(\nu)$ and $G(\lambda,\mu)$ are normalized, i.e.

$$\int_0^\infty \int_0^\infty G(\lambda, \mu)\, d\lambda\, d\mu = 1 \tag{36}$$

$$\int_{-\infty}^{+\infty} F(\nu)\, d\nu = 1\ , \tag{37}$$

we can show that W is normalized over Λ:

$$\int_{-\infty}^{+\infty} d\tau\, \nu\, Q_0(\nu\tau - \nu t) \int_{-\infty}^{+\infty} d\nu\, F(\nu) \int_{-\pi}^{+\pi} d\alpha \int_{-\pi}^{+\pi} d\beta\, \frac{1}{4\pi^2} \int_0^\infty d\lambda \int_0^\infty d\mu\, G(\lambda, \mu) = \int_{-\infty}^{+\infty} ds\, Q_0(s - s_0) = Nq\ .$$

Setting

$$\rho_t(x, z) = \frac{1}{\pi^2} \int_{\frac{x}{\sqrt{\beta_x}}}^{\infty} \int_{\frac{z}{\sqrt{\beta_z}}}^{\infty} \frac{G(\lambda, \mu)}{\sqrt{\lambda^2\beta_x - x^2}\,\sqrt{\mu^2\beta_z - z^2}}\, d\lambda\, d\mu\ , \tag{38}$$

we obtain with Eqs. (33), (34) and (35) for ρ and j the expressions

$$\rho(\vec{x}, t) = Q(s, t)\, \rho_t(x, z) \tag{39}$$

$$\vec{j}(\vec{x}, t) = J(s, t)\, \rho_t(x, z)\ , \tag{40}$$

where

$$Q(s, t) = \int_{-\infty}^{+\infty} Q_0(s - \nu t)\, F(\nu)\, d\nu \tag{41}$$

$$J(s, t) = \int_{-\infty}^{+\infty} Q_0(s - \nu t)\, \nu F(\nu)\, d\nu \tag{42}$$

are the current and charge per unit length at time t, which have the initial values at t = 0:

$$\begin{aligned} Q(s, o) &= Q_0(s) \\ J(s, o) &= Q_0(s)\, \hat{\nu} \ , \end{aligned} \tag{43}$$

where $\hat{\nu}$ denotes the average velocity in the s-direction:

$$\hat{\nu} \int_{-\infty}^{+\infty} F(\nu)\nu\, d\nu \ .$$

The relations (41) and (42) can be used to calculate the longitudinal dispersion of a coasting bunch due to the initial velocity distribution of its particles.

Let us return to the transverse density distribution. If we introduce the local amplitudes

$$\begin{aligned} a &= \lambda\sqrt{\beta_x} \\ b &= \mu\sqrt{\beta_z} \end{aligned} \tag{44}$$

and the corresponding distribution function

$$g(a, b) = \frac{G(\lambda, \mu)}{\sqrt{\beta_x \beta_z}} \ , \tag{45}$$

which, due to Eq. (36), has also to satisfy the normalization

$$\int_0^{\infty}\int_0^{\infty} g(a, b)\, da\, db = 1 \ , \tag{46}$$

we can simplify expression (38):

$$\rho_t(x, z) = \frac{1}{\pi^2}\int_x^{\infty}\int_z^{\infty} \frac{g(a, b)}{\sqrt{a^2 - x^2}\,\sqrt{b^2 - z^2}}\, da\, db \ . \tag{47}$$

Using the symbols introduced in Section 2 and the normalization theorem (22) we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_t(x, z)\, dx\, dz = \frac{4}{\pi^2}\int_0^{\infty}\int_0^{\infty} K(x, a)\, K(z, b)\, g(a, b) \quad dx\, dz = 1 \ . \tag{48}$$

Under the restriction that $\rho_t(x,z)$ is differentiable throughout the whole (x,z)-plane, the inversion of Eq. (47) can be performed with formula (24) and yields

$$g(a,b) = 4ab\ K(a,x)\ K(b,z)\ \frac{\partial^2 \rho_t(x,z)}{\partial x \partial z}\ , \tag{49}$$

Reinserting λ, μ we get for $G(\lambda,\mu)$

$$G(\lambda,\mu) = 4\lambda\mu\beta_x\beta_z\ K(\lambda\sqrt{\beta_x}, x)\ K(\mu\sqrt{\beta_z}, z)\ \frac{\partial^2 \rho_t(x,z)}{\partial x \partial z}\ . \tag{50}$$

In cases where ρ_t is only differentiable inside a finite domain, the formulae (49) and (50) are invalid, but the inversion can also be performed using a more general inversion formula (see Appendix V).

4. THE NORMALIZED TRANSVERSE PHASE SPACES AND THEIR PROJECTIONS

For practical calculations, one preferably uses simpler phase spaces than the real phase space (μ-space) of the canonical conjugate variables. In our case, where the motion is given by Eqs. (23), we have two possibilities to simplify the phase-space representation. We either keep unchanged the constant of the motion λ, μ or the real space variables x, z.

i) The substitutions

$$\xi = \frac{x}{\sqrt{\beta_x}}\ , \quad \zeta = \frac{z}{\sqrt{\beta_z}} \tag{51}$$

reduce Eqs. (23) to

$$\xi = \lambda \cos(\psi_x + \alpha)$$
$$\zeta = \mu \cos(\psi_z + \beta)$$

and the particles can be regarded as harmonic oscillators with respect to their phases ψ_x, ψ_z. Since we have

$$\xi' = \frac{d\xi}{d\psi_x} = -\sqrt{\lambda^2 - \xi^2}$$
$$\zeta' = \frac{d\zeta}{d\psi_z} = -\sqrt{\mu^2 - \zeta^2}\ ,$$

the trajectories in the phase planes (ξ,ξ'), (ζ,ζ') are circles with radii λ and μ, respectively. Owing to the assumption that all values of α, β are equally probable (i.e. incoherence of the betatron oscillations), we have a constant population along the circumference of each circle. Thus all the information of the transverse phase space lies in the function $G(\lambda,\mu)$ and we obtain the distribution for the four-dimensional "quasi" phase-space $\{\xi,\zeta,\xi',\zeta'\}$ by substituting

$$\lambda = \sqrt{\xi^2 + \xi'^2} \qquad \mu = \sqrt{\zeta^2 + \zeta'^2} \tag{52}$$

in $G(\lambda,\mu)$. If we measure the transverse density $\rho_t(x,z)$ and introduce a normalized density $\rho^*(\xi,\zeta)$:

$$\rho^*(\xi,\zeta) = \rho_t(\xi\sqrt{\beta_x}, \zeta\sqrt{\beta_z}) \sqrt{\beta_x\beta_z} \tag{53}$$

(which fulfils the normalization

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho^* \, d\xi \, d\zeta = 1$$

owing to the normalization of ρ_t), we obtain $G(\lambda,\mu)$ by

$$G(\lambda,\mu) = 4\lambda\mu \, K(\lambda,\xi) \, K(\mu,\zeta) \, \frac{\partial^2\rho^*}{\partial\xi\partial\zeta} \ . \tag{54}$$

ii) The second possibility for a simplified phase space lies in the substitution (44). Introducing the new phase-space variables

$$x' := \sqrt{\beta_x}\,\xi' \qquad z' := \sqrt{\beta_z}\,\zeta' \ , \tag{55}$$

the particles move in the "normalized" phase space $\{x,z,x',z'\}$ along the circles

$$x^2 + x'^2 = a^2 \qquad z^2 + z'^2 = b^2 \ . \tag{56}$$

The relation between the amplitude distribution $g(a,b)$, defined in Eq. (45), and the transverse density $\rho_t(x,z)$ is given by Eq. (49). But it must be pointed out that x' and z' have not the meaning of a real derivative, neither with respect to τ or to s, nor to the phases ψ_x, ψ_z! Nevertheless, this phase space is the most convenient one for practical use.

In Appendix III, Table III.2 gives the relation between the functions $\rho_t(x,z)$ and $g(a,b)$ and their projections on the subspaces $P(x,x')$, $p(x)$ and $n(a)$, which can easily be obtained via the normalization theorem (21). The relations between $G(\lambda,\mu)$, $\rho^*(\xi,\zeta)$ and their projections are tabulated in Table III.3. Some particular distributions are listed in Table III.4

5. SURFACE AND RADIAL DENSITIES IN ROTATIONALLY-SYMMETRIC SYSTEMS AND THEIR ONE-DIMENSIONAL PROJECTION

In Section 1 we had the relation

$$p(x) = 2\int_x^R \frac{P(r)r\,dr}{\sqrt{r^2 - x^2}} = 2K(x, r)\,rP(r)$$

between the surface density P(r) of a physical quantity [which is rotationally-symmetric around the origin of the (x,y)-plane and therefore a function of $r = (x^2 + y^2)^{\frac{1}{2}}$] and its projection p(x) onto the x-axis. P(r) is assumed to be normalized:

$$\iint\limits_{x^2+y^2\leq R^2} P(r)\,dx\,dy = 1 \ .$$

Changing to polar coordinates dx dy = r dr dϕ and integrating over ϕ yields

$$2\pi\int_0^R P(r)r\,dr = 1 \ . \qquad (57)$$

Using Eq. (21) we get from Eq. (3):

$$\int_{-R}^{+R} p(x)\,dx = 4\int_0^R K(x,r)r\,P(r)\,dx = 2\pi\int_0^R rP(r)\,dr = 1$$

as expected, since the total quantity must be independent of the way of integration! Instead of the surface density P(r) we frequently use the radial density n(r), defined as

$$n(r) = 2\pi r\,P(r) \qquad (58)$$

If we substitute this in Eq. (3) we obtain

$$p(x) = \frac{1}{\pi}\int_x^R \frac{n(r)}{\sqrt{r^2 - x^2}}\,dr \qquad (59)$$

and the inversion yields

$$n(r) = -2\frac{d}{dr}\int_r^R \frac{p(x)x}{\sqrt{x^2 - r^2}}\,dx = -2r\int_r^R \frac{p'(x)}{\sqrt{x^2 - r^2}}\,dx$$

with p'(x) = dp/dx. These relations, together with some examples, are tabulated in Appendix IV.

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APPENDIX I

INVERSION OF g(x) = K(x,t)f(t) AND h(x) = J(x,t)f(t) WITH THE ERDELYI-KOBER OPERATORS

The Erdélyi-Kober operators are defined by

$$K_{\alpha,\beta}f(t) = \frac{2x^{2\alpha}}{\Gamma(\beta)} \int_x^R (t^2 - x^2)^{\beta-1}\, t^{-2\alpha-2\beta+1}\, f(t)\, dt \tag{I.1}$$

$$J_{\alpha,\beta}f(t) = \frac{2x^{-2\alpha-2\beta}}{\Gamma(\beta)} \int_0^x (x^2 - t^2)^{\beta-1}\, t^{2\alpha+1}\, f(t)\, dt \tag{I.2}$$

for $\alpha > -\frac{1}{2}$, $\beta > 0$, and for $-1 < \beta < 0$ by

$$K_{\alpha,\beta}f(t) = -\frac{x^{2\alpha-1}}{\Gamma(1+\beta)} \frac{d}{dx} \int_x^R t^{-2\alpha-2\beta+1}(t^2 - x^2)^{\beta}\, f(t)\, dt \tag{I.3}$$

$$J_{\alpha,\beta}f(t) = \frac{x^{-2\alpha-2\beta-1}}{\Gamma(1+\beta)} \frac{d}{dx} \int_0^x t^{2\alpha+1}(x^2 - t^2)^{\beta}\, f(t)\, dt\ . \tag{I.4}$$

If we let β tend to zero, we obtain the identity

$$K_{\alpha,0} = J_{\alpha,0} = I\ ,$$

where I stands for the identity operator. The following product rules

$$J_{\alpha,\beta}\, J_{\alpha+\beta,\gamma} = J_{\alpha,\beta+\gamma} \tag{I.5}$$

$$K_{\alpha,\beta}\, K_{\alpha+\beta,\gamma} = K_{\alpha,\beta+\gamma} \tag{I.6}$$

can be shown by interchanging the order of integrations and making use of the formulae

$$2\int_u^x (x^2 - t^2)^{\beta-1} (t^2 - u^2)^{\gamma-1}\, t^{1-2\beta-2\gamma}\, dt = \frac{\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\, u^{-2\beta}\, x^{-2\gamma}(x^2 - u^2)^{\beta+\gamma-1}$$

for the calculation of the inner integral.

The inversion follows now formally from

$$K^{-1}_{\alpha,\beta} = K_{\alpha+\beta,-\beta} \tag{I.7}$$

$$J^{-1}_{\alpha,\beta} = J_{\alpha+\beta,-\beta}\ . \tag{I.8}$$

These operators can also be obtained from the operators (6) and (7) given by Srivastav[10], but have been derived before in connection with fractional integration and dual integral transforms by Erdélyi[6], Erdélyi and Kober[6,7] and Sneddon[8,9]. Since we know already the inverses of

$$K(x,t) = \int_x^R dt \, \frac{1}{\sqrt{t^2 - x^2}} \quad \text{and} \quad J(x,t) = \int_0^x dt \, \frac{1}{\sqrt{x^2 - t^2}}$$

we see that the restriction $\alpha > -\frac{1}{2}$ can be relaxed at least for $J_{\alpha,\beta}$. Doing this, the inversion of Eqs. (4) and (5) can be formally obtained by comparing Eqs. (4) and (I.1), (5) and (I.2)

$$K(x,t) = \frac{\sqrt{\pi}}{2} K_{0,\frac{1}{2}}$$

$$J(x,t) = \frac{\sqrt{\pi}}{2} J_{-\frac{1}{2},\frac{1}{2}}$$

which yields the inversion formulae

$$K^{-1}(x,t) = \frac{2}{\sqrt{\pi}} K_{+\frac{1}{2},-\frac{1}{2}} = -\frac{2}{\pi} \frac{d}{dt} K(x,t)t$$

$$J^{-1}(x,t) = \frac{2}{\sqrt{\pi}} J_{0,-\frac{1}{2}} = \frac{2}{\pi} \frac{d}{dt} J(x,t)t \ .$$

APPENDIX II

COLLECTION OF THE MOST IMPORTANT FORMULAE CONCERNING THE TRANSFORMATION $g(x) = K(x,t)f(t)$

Definition:

$$K(x,t) := \int_{x}^{R} \frac{dt}{\sqrt{t^2 - x^2}}$$

Inverse:

$$K(t,x)^{-1} \cdot K(x,t) = I$$

$$K(t,x)^{-1} = -\frac{2}{\pi}\frac{d}{dt} K(t,x)x = -\frac{2}{\pi}\, t\, K(t,x)\frac{d}{dx}$$

Normalization theorem:

$$\int_{0}^{R} K(x,t)\, f(t)\, dx = \frac{\pi}{2}\int_{0}^{R} f(t)\, dt$$

Special properties of the operator:

$$K(x,t)\frac{d}{dt} = \frac{1}{x}\frac{d}{dx} K(x,t)t\,,$$

$$\frac{d}{dx} K(x,t) = xK(x,t)\frac{d}{dt}\frac{1}{t}$$

$$K(\alpha x,t) = \int_{\alpha x}^{\alpha R} \frac{dt}{\sqrt{t^2 - (\alpha x)^2}} = \int_{x}^{R} \frac{d(t/\alpha)}{\sqrt{(t/\alpha)^2 - x^2}} = K(x, t/\alpha)\ .$$

Transformation pairs

$f(t) \quad 0 \le t \le R$	$g(x) = K(x,t)f(t) \quad 0 \le x \le R$
A. General rules $\alpha f_1 + \beta f_2$	$\alpha g_1 + \beta g_2$ Linearity
$tf(t)$	$f(R)\sqrt{R^2 - x^2} + x^2 K(x,t)\,\frac{df}{dt} - K(x,t)t^2\,\frac{df}{dt}$ (Partial integration)
B. Finite range $(R < \infty)$	
1	$\operatorname{arcosh} \frac{R}{x}$
t	$\sqrt{R^2 - x^2}$
t^2	$\frac{1}{2}\left(R\sqrt{R^2 - x^2} + x^2 \operatorname{arcosh} \frac{R}{x}\right)$
t^3	$\frac{1}{3}(R^2 + 2x^2)\sqrt{R^2 - x^2}$
$t^n \quad n \ge 2$	$\frac{1}{n}\left[R^{n-1}\sqrt{R^2 - x^2} + (n-1)x^2 K(x,t)t^{n-2}\right]$
$\frac{1}{t}$	$\frac{1}{x}\left(\pi/2 - \arcsin \frac{x}{R}\right)$
$f(t) = \begin{cases} \frac{2}{\pi}\frac{t}{\sqrt{a^2 - t^2}} & 0 \le t < a \\ 0 & a \le t \le R \end{cases}$	$g(x) = \begin{cases} 1 & 0 \le x < a \\ 0 & a \le x \le R \end{cases}$
$f(t) = \begin{cases} \frac{2}{\pi} kt \operatorname{arcosh} \frac{a}{kt} & 0 \le t \le \frac{a}{k} \\ 0 & \frac{a}{k} < t \le R \end{cases}$	$g(x) = \begin{cases} a - kx & 0 \le x \le \frac{a}{k} \\ 0 & \frac{a}{k} < x \le R \end{cases}$
$f(t) = \begin{cases} \frac{4}{\pi} t \sqrt{a^2 - t^2} & 0 \le t \le a \\ 0 & a < t \le R \end{cases}$	$g(x) = \begin{cases} a^2 - x^2 & 0 \le x \le a \\ 0 & a < x \le R \end{cases}$

Transformation pairs (cont.)

$f(t) = \begin{cases} \sum_{k=0}^{N} c_{2k+1} t^{2k+1} & 0 \le t \le 1 \text{ *)} \\ 0 & 1 < t \le R \end{cases}$	$g(x) = \begin{cases} \left(\sum_{k=0}^{N} b_{2k} x^{2k} \right) \sqrt{1 - x^2} = \text{ *)} \\ = \sqrt{1 - x^2} \sum_{k=0}^{N} c_{2k+1} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} \times \\ \times \frac{(-1)^m}{2\ell + 1} \left(x^2\right)^{k+m-\ell} \quad 0 \le x \le 1 \\ 0 \quad 1 < x \le R \end{cases}$
$f(t) = \begin{cases} t(1 - t^2)^N & 0 \le t \le 1 \\ & N = 0,1,2,\ldots \\ 0 & 1 < t \le R \end{cases}$	$g(x) = \begin{cases} \sqrt{1 - x^2} \sum_{k=0}^{N} \binom{N}{k} (-1)^k \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{1}{2\ell + 1} \times \\ \times \sum_{m=0}^{t} \binom{\ell}{m} (-1)^m \left(x^2\right)^{k+m-\ell} \quad 0 \le x \le 1 \\ 0 \quad 1 < x \le R \end{cases}$

*) The coefficients are related by (see Bovet, Ref. 2):

$$\begin{pmatrix} b_0 \\ b_2 \\ b_4 \\ b_6 \\ b_8 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots \\ 0 & \frac{2}{3} & \frac{4}{15} & \frac{6}{35} & \frac{8}{63} & \cdots \\ 0 & 0 & \frac{8}{15} & \frac{8}{35} & \frac{16}{105} & \cdots \\ 0 & 0 & 0 & \frac{16}{35} & \frac{64}{315} & \cdots \\ 0 & 0 & 0 & 0 & \frac{128}{315} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \\ c_5 \\ c_7 \\ c_9 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Transformation pairs (cont.)

$f(t) \qquad 0 \leq t < \infty$	$g(x) = K(x,t)f(t) \qquad 0 \leq x < \infty$
C. Infinite range $(R = \infty)$ $2t\, e^{-\mu t^2} = H_1(t)\, e^{-\mu t^2}$	$\sqrt{\frac{\pi}{\mu}}\, e^{-\mu x^2}$
$(8t^3 - 12t)\, e^{-\mu t^2} = H_3(t)\, e^{-\mu t^2}$	$\left(4x^2 - 6 + \frac{2}{\mu}\right) \sqrt{\frac{\pi}{\mu}}\, e^{-\mu x^2}$
$(32t^5 - 160t^3 + 120t)\, e^{-\mu t^2} =$ $= H_5(t)\, e^{-\mu t^2}$	$\left(16x^4 + \frac{16}{\mu} x^2 - 80x^2 + 60 - \frac{40}{\mu} + \frac{12}{\mu^2}\right) \times$ $\times \sqrt{\frac{\pi}{\mu}}\, e^{-\mu x^2}$
$H_{2k+1}(t)\, e^{-\mu t^2}$	$\Psi_{2k+1}(x,\mu) \sqrt{\frac{\pi}{\mu}}\, e^{-\mu x^2}$ †)
$4t \sqrt{\frac{\mu}{\pi}} \left[\Psi_{2k}(t,\mu) - 2\frac{k}{\mu} \Psi_{2k-1}(t,\mu)\right] \times$ $\times\, e^{-\mu t^2}$ †)	$H_{2k}(x)\, e^{-\mu x^2}$
$t^{-2n} \qquad n = 1,2,\ldots$	$\frac{(n-1)!\,2^{n-1}}{(1;2;n)x^{2n}} \qquad n = 1,2,\ldots$ ††)

†) $$\Psi_{2k}(x,\mu) = \frac{1}{2}(2k)! \sum_{s=0}^{k} \frac{(4x^2)^s}{s!} \sum_{\ell=0}^{k-s} \frac{(-1)^{k-s-\ell}\,(s+\ell)!\,(2\ell-1)!!}{(k-s-\ell)!\,\ell!\,(2s+2\ell)!} \left(\frac{2}{\mu}\right)^{\ell}$$

$$\Psi_{2k+1}(x,\mu) = 2^k k!\,(2k+1)!! \sum_{s=0}^{k} \frac{(2x^2)^s}{s!} \sum_{\ell=0}^{k-s} \frac{(-1)^{k-s-\ell}\,(2\ell-1)!!}{(k-s-\ell)!\,\ell!\,(2s+2\ell+1)!!} \frac{1}{\mu^{\ell}}$$

††) $$(a;b;c) = \frac{b^c\,\Gamma(a/b + c)}{\Gamma(a/b)}$$

APPENDIX III

RELATIONS BETWEEN THE DENSITY DISTRIBUTIONS IN THE DIFFERENT SPACES

Table III.1

Variables (Definitions)	Range	Distribution
Real transverse space coordinates x,z	$-\infty < x < +\infty$ $-\infty < z < +\infty$	$\rho(x,z)$, $p(x)$ *)
$x' := \sqrt{\beta_x}\,\frac{d}{d\Psi_x}\left(\frac{x}{\sqrt{\beta_x}}\right)$ $z' := \sqrt{\beta_z}\,\frac{d}{d\Psi_z}\left(\frac{z}{\sqrt{\beta_z}}\right)$	$-\infty < x' < +\infty$ $-\infty < z' < +\infty$	$F(x,x',z,z')$, $P(x,x')$
Local amplitudes $a^2 = x^2 + x'^2$, $b^2 = z^2 + z'^2$	$0 \leq a < +\infty$ $0 \leq b < +\infty$	$g(a,b)$, $n(a)$
Normalized transverse space coordinates $\xi = x/\sqrt{\beta_x}$, $\zeta = z/\sqrt{\beta_z}$	$-\infty < \xi < +\infty$ $-\infty < \zeta < +\infty$	$\rho^*(\zeta,\eta) = \rho\sqrt{\beta_x\beta_z}$ $p^*(\xi) = p\sqrt{\beta_x}$
$\xi' = \frac{d\xi}{d\Psi_x}$, $\zeta' = \frac{d\zeta}{d\Psi_z}$ $\xi' = x'/\sqrt{\beta_x}$, $\zeta' = z'/\sqrt{\beta_z}$	$-\infty < \xi' < +\infty$ $-\infty < \zeta' < +\infty$	$F^*(\xi,\xi',\zeta,\zeta') = F\beta_x\beta_z$ $P(\xi,\xi') = P\beta_x$
Normalized amplitudes $\lambda = a/\sqrt{\beta_x}$, $\mu = b/\sqrt{\beta_z}$ $\lambda^2 = \xi^2 + \xi'^2$, $\mu^2 = \zeta^2 + \zeta'^2$	$0 \leq \lambda < +\infty$ $0 \leq \mu < +\infty$	$G(\lambda,\mu) = g\sqrt{\beta_x\beta_z}$ $n^*(\lambda) = n\sqrt{\beta_x}$

λ, μ are constants of the motion and related to the emittance ε by $\varepsilon = \lambda_{max}\mu_{max}\,\pi$.
The related densities G and n^* are also invariants.

*) ρ in these tables is identical with $\rho_t(x,z)$ in the text.

Table III.2

	$F(x,z,x',z')$	$g(a,b)$	$\rho(x,z)$
$F =$		$\frac{1}{4\pi^2 ab}\, g(a,b)$	$\frac{1}{\pi^2}\, K(a,x)K(b,z)\, \frac{\partial^2\rho}{\partial x\partial z}$ *)
$g =$	$4\pi^2 abF$		$4abK(a,x)K(b,z)\, \frac{\partial^2\rho}{\partial x\partial z}$ *)
$\rho =$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F\, dx'\, dz'$	$\frac{1}{\pi^2}\, K(x,a)K(z,b)g(a,b)$ *)	
	$P(x,x') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F\, dz\, dz'$	$n(a) = \int_0^{\infty} g(a,b)\, db$	$p(x) = \int_{-\infty}^{+\infty} \rho(x,z)\, dz$
$P =$		$\frac{1}{2\pi a}\, n(a)$	$-\frac{1}{\pi}\, K(a,x)\, \frac{dp}{dx}$
$n =$	$2\pi aP$		$-2aK(a,x)\, \frac{dp}{dx}$
$p =$	$\int_{-\infty}^{+\infty} P\, dx'$	$\frac{1}{\pi}\, K(x,a)n(a)$	

Table III.3

	$F^*(\xi,\zeta,\xi',\zeta')$	$G(\lambda,\mu)$	$\rho^*(\xi,\zeta)$
$F^* =$		$\frac{1}{4\pi^2\lambda\mu}\, G(\lambda,\mu)$	$\frac{1}{\pi^2}\, K(\lambda,\xi)K(\mu,\zeta)\, \frac{\partial^2\rho^*}{\partial\xi\partial\zeta}$ *)
$G =$	$4\pi^2\lambda\mu F$		$4\lambda\mu K(\lambda,\xi)K(\mu,\zeta)\, \frac{\partial^2\rho^*}{\partial\xi\partial\zeta}$ *)
$\rho^* =$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F^*\, d\xi'\, d\zeta'$	$\frac{1}{\pi^2}\, K(\xi,\lambda)K(\zeta,\mu)G(\lambda,\mu)$ *)	
	$P^*(\xi,\xi') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F^*\, d\zeta\, d\zeta'$	$n^*(\lambda) = \int_0^{\infty} G(\lambda,\mu)\, d\mu$	$p^*(\xi) = \int_{-\infty}^{+\infty} \rho^*(\xi,\zeta)\, d\zeta$
$P^* =$		$\frac{1}{2\pi\lambda}\, n^*(\lambda)$	$-\frac{1}{\pi}\, K(\lambda,\xi)\, \frac{dp^*}{d\xi}$
$n^* =$	$2\pi\lambda P^*$		$-2\lambda K(\lambda,\xi)\, \frac{dp^*}{d\xi}$
$p^* =$	$\int_{-\infty}^{+\infty} P^*\, d\xi'$	$\frac{1}{\pi}\, K(\xi,\lambda)n^*(\lambda)$	

*) For finite domains of differentiability see the general inversion formulae of Appendix V.

SOME NORMALIZED TWO-DIMENSIONAL DISTRIBUTIONS AND THEIR PROJECTION

Table III.4

	Gaussian	Constant over rectangular domain	Kapchinskij-Vladimirskij normal distribution[12]	Parabolic
$\rho(x,z)$	$\frac{1}{2\pi AB} e^{-\frac{x^2}{2A^2} - \frac{z^2}{2B^2}}$	$\frac{1}{4AB}$ $\|x\| < A$, $\|z\| < B$ 0 $\|x\| \geq A$, $\|z\| \geq B$	$\frac{1}{\pi AB}$ $\frac{x^2}{A^2} + \frac{z^2}{B^2} < 1$ 0 $\frac{x^2}{A^2} + \frac{z^2}{B^2} > 1$	$\frac{2}{\pi AB}\left(1 - \frac{x^2}{A^2} - \frac{z^2}{B^2}\right)$ $\frac{x^2}{A^2} + \frac{z^2}{B^2} \leq 1$ 0 $\frac{x^2}{A^2} + \frac{z^2}{B^2} > 1$
$g(a,b)$	$\frac{ab}{A^2B^2} e^{-\frac{a^2}{2A^2} - \frac{b^2}{2B^2}}$	$\frac{ab}{AB} \frac{1}{\sqrt{(A^2 - a^2)(B^2 - b^2)}}$ $a < A, \; b < B$ 0 $a \geq A, \; b \geq B$	$\frac{ab}{AB} \delta\left[1 - \frac{a^2}{A^2} - \frac{b^2}{B^2}\right]$	$\frac{8ab}{A^2B^2}$ $\frac{a^2}{A^2} + \frac{b^2}{B^2} \leq 1$ 0 $\frac{a^2}{A^2} + \frac{b^2}{B^2} > 1$
$p(x)$	$\frac{1}{\sqrt{2\pi}A} e^{-\frac{x^2}{2A^2}}$	$\frac{1}{2A}$ $\|x\| < A$ 0 $\|x\| \geq A$	$\frac{2}{\pi A^2}\sqrt{A^2 - x^2}$ $\|x\| < A$ 0 $\|x\| > A$	$\frac{8}{3\pi A^4}(A^2 - x^2)^{3/2}$ $\|x\| \leq A$ 0 $\|x\| > A$
$n(a)$	$\frac{a}{A^2} e^{-\frac{a^2}{2A^2}}$	$\frac{a}{A}\frac{1}{\sqrt{A^2 - a^2}}$ $a < A$ 0 $a \geq A$	$\frac{2a}{A^2}$ $a < A$ 0 $a > A$	$\frac{4a}{A^4}(A^2 - x^2)$ $a \leq A$ 0 $a > A$

APPENDIX IV

RELATIONS BETWEEN SURFACE, RADIAL, AND PROJECTED DENSITIES FOR ROTATIONALLY-SYMMETRIC SYSTEMS

Table IV.1

	Surface density P(r)	Radial density n(r)	Projected density p(x)
P =		$\frac{1}{2\pi r}\, n(r)$	$-\frac{1}{\pi} K(r,x) \frac{dp}{dx}$
n =	$2\pi r P(r)$		$-\ 2rK(r,x) \frac{dp}{dx}$
p =	$2K(x,r) r P(r)$	$\frac{1}{\pi} K(x,r) n(r)$	

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P\, dx\, dy = 1 \qquad \int_{0}^{+\infty} n\, dr = 1 \qquad \int_{-\infty}^{+\infty} p\, dx = 1$$

SOME NORMALIZED DENSITY DISTRIBUTIONS

Table IV.2

	P(r)		p(x)	
Gaussian	$\frac{\mu}{\pi} e^{-\mu r^2}$		$\sqrt{\frac{\mu}{\pi}}\, e^{-\mu x^2}$	
Constant	$\frac{1}{\pi R^2}$	$r \le R$	$\frac{2}{\pi R^2} \sqrt{R^2 - x^2}$	$\lvert x\rvert \le R$
	0	$r > R$	0	$\lvert x\rvert > R$
Parabolic	$\frac{2}{\pi R^4} (R^2 - r^2)$	$r \le R$	$\frac{8}{3\pi R^4} \left(R^2 - x^2\right)^{3/2}$	$\lvert x\rvert \le R$
	0	$r > R$	0	$\lvert x\rvert > R$
Conical	$\frac{3}{R^2 \pi} \left(1 - \frac{r}{R}\right)$	$r \le R$	$\frac{1}{2} \left(\sqrt{R^2 - x^2} - \frac{x^2}{R} \operatorname{arcosh} \frac{R}{\lvert x\rvert}\right)$	$\lvert x\rvert \le R$
	0	$r > R$	0	$\lvert x\rvert > R$
Constant projected density	$\frac{1}{2\pi A} \frac{1}{\sqrt{A^2 - r^2}}$	$r < A$	$\frac{1}{2A}$	$\lvert x\rvert < A$
	0	$r \ge A$	0	$\lvert x\rvert \ge A$

APPENDIX V

THE TWO-DIMENSIONAL OPERATOR $\overline{K(x,t)K(z,u)}_D$ AND ITS INVERSE IN A MORE GENERAL CASE

For the inversion formulae (49) and (54), we assumed the functions $\rho(x,z)$ and $\rho^*(\xi,\zeta)$ to be defined and differentiable throughout the infinite domains $\{x,z|0 \leq x < +\infty,\ 0 \leq z < +\infty\}$ and $\{\xi,\zeta|0 \leq \xi < +\infty,\ 0 \leq \zeta < +\infty\}$, respectively. Unfortunately, this assumption cannot be made for most practical cases, where the distribution is set to zero outside a bounded area D. In these cases we must restrict the domain of integration to D; otherwise the discontinuity of the first partial differentials at the boundary would prevent an inversion. For these cases the following generalization holds.

Let D be a compact domain, defined by

$$D = \{t, u|t \geq 0,\ u \geq 0,\ B(u,t) \geq 0\}\ , \qquad (V.1)$$

where $B(u,t) = 0$ represents the boundary of D. We shall further assume that the equations

$$B[\phi(u), u] = 0$$
$$B[t, \psi(t)] = 0 \qquad (V.2)$$

have unique solutions $\phi(u)$, $\Psi(t)$, which are at least once differentiable functions for $t,u \geq 0$. We define the two-dimensional operator

$$\overline{K(x,t)\ K(z,u)}_D := \iint\limits_{D(x,z)} \frac{dt\ du}{\sqrt{(t^2 - x^2)(u^2 - z^2)}}\ , \qquad (V.3)$$

where $D'(x,t)$ denotes the subset

$$D(x,z) = \{t, u|t \geq x,\ u \geq z,\ B(t,u) \geq 0\} \subset D\ .$$

Let us now assume that $F(t,u)$ is a regular, bounded real function defined over D. Under these restrictions $G(x,z)$, with

$$G(x,z) = \overline{K(x,t)\ K(z,u)}_D\ F(t,u)\ . \qquad (V.4)$$

is also a regular function over D, which equals the integrals

$$G(x, z) = \int_{x}^{\phi(z)} \frac{dt}{\sqrt{t^2 - x^2}} \int_{z}^{\psi(t)} \frac{du\, F(t, u)}{\sqrt{u^2 - z^2}} \tag{V.5a}$$

$$G(x, z) = \int_{z}^{\psi(x)} \frac{du}{\sqrt{u^2 - z^2}} \int_{x}^{\phi(u)} \frac{dt\, F(t, u)}{\sqrt{t^2 - x^2}} \, , \tag{V.5b}$$

where the outer right integration has to be performed first.

The integral

$$A(t, z) = \int_{z}^{\psi(t)} \frac{du\, F(t, u)}{\sqrt{u^2 - z^2}} \tag{V.6}$$

can be regarded as a function of t, which depends on the parameter z. Thus, G(x,z) represents its one-dimensional Abel-type transform

$$G(x, z) = \int_{x}^{\phi(z)} \frac{dt\, A(t, z)}{\sqrt{t^2 - x^2}} \tag{V.7}$$

on the finite domain $0 \le x \le \phi(z)$. Since the dependency of the upper limit on a parameter does not affect the validity of the inversion formula (14), we obtain for A(t,z) the expressions

$$A(t, z) = -\frac{2}{\pi} \frac{d}{dt} \int_{t}^{\phi(z)} \frac{dx\, xG(x, z)}{\sqrt{x^2 - t^2}} = -\frac{2}{\pi}\, t \int_{t}^{\phi(z)} \frac{dx}{\sqrt{x^2 - t^2}} \frac{\partial G(x, z)}{\partial x} \, . \tag{V.8}$$

In the same way, we can invert Eq. (V.6) with z,u as the pair of transformed variables and obtain

$$F(t, u) = -\frac{2}{\pi} \int_{u}^{\psi(t)} \frac{dz\, zA(t, z)}{\sqrt{z^2 - u^2}} = -\frac{2}{\pi}\, u \int_{u}^{\psi(t)} \frac{dz}{\sqrt{z^2 - u^2}} \frac{\partial A(t, z)}{\partial z} \, . \tag{V.9}$$

If we now insert (V.8) in the right-hand side of Eq. (V.9) we get

$$F(t, u) = \frac{4}{\pi^2}\, tu \int_{u}^{\psi(t)} \frac{dz}{\sqrt{z^2 - u^2}} \left[\frac{\partial}{\partial z} \int_{t}^{\phi(z)} \frac{dx}{\sqrt{x^2 - t^2}} \frac{\partial G(x, z)}{\partial x} \right] . \tag{V.10a}$$

In an analogous way we can show the validity of the equation

$$F(t, u) = \frac{4}{\pi^2}\, tu \int_{t}^{\phi(u)} \frac{dx}{\sqrt{x^2 - t^2}} \left[\frac{\partial}{\partial z} \int_{u}^{\psi(x)} \frac{dz}{\sqrt{z^2 - u^2}} \frac{\partial G(x, z)}{\partial z} \right] . \tag{V.10b}$$

The differentiation of the right-hand side integral of Eq. (V.10a) yields

$$\frac{\partial}{\partial z}\int_t^{\phi(z)} \frac{dx}{\sqrt{x^2-t^2}}\,\frac{\partial G(x,z)}{\partial x} = \int_t^{\phi(z)} \frac{dx}{\sqrt{x^2-t^2}}\,\frac{\partial^2 G(x,z)}{\partial z\,\partial x} + \frac{1}{\sqrt{\phi^2-t^2}}\left[\frac{\partial G(x,z)}{\partial x}\right]_{x=\phi(z)}\frac{d\phi}{dz} \qquad \text{(V.11)}$$

and F(t,u) can be calculated with the formulae

$$F(t,u) = \frac{4}{\pi^2}\,tu \iint_{D(t,u)} \frac{dx\,dz}{\sqrt{(x^2-t^2)(z^2-u^2)}}\,\frac{\partial^2 G(x,z)}{\partial z\,\partial x}$$

$$+\frac{4}{\pi^2}\,tu\int_u^{\psi(t)} \frac{dz}{\sqrt{z^2-u^2}}\,\frac{1}{\sqrt{\phi^2(z)-t^2}}\left[\frac{\partial G(x,z)}{\partial x}\right]_{x=\phi(z)}\frac{d\phi}{dz}\,, \qquad \text{(V.12a)}$$

where D(t,u) denotes the domain

$$D(t,u) = \{x,z \mid t \le x,\ u \le z,\ B(x,z) \ge 0\}\ .$$

The substitution $x = \phi(z)$ as independent variable in the second integral of Eq. (V.12a) yields the equality

$$\int_u^{\psi(t)} \frac{dz}{\sqrt{z^2-u^2}}\,\frac{1}{\sqrt{\psi^2(z)-t^2}}\left[\frac{\partial G(x,z)}{\partial x}\right]_{x=\phi(z)}\frac{d\phi}{dz}$$

$$= -\int_t^{\phi(u)} \frac{dx}{\sqrt{x^2-t^2}}\,\frac{1}{\sqrt{\psi^2(x)-u^2}}\left[\frac{\partial G(x,z)}{\partial x}\right]_{z=\psi(x)} \qquad \text{(V.13)}$$

where the identities

$$\phi[\psi(t)] = t \qquad \psi[\phi(u)] = u \qquad \text{(V.14)}$$

following from Eq. (V.2) have been used. With the definition (V.3) and formula (V.13) we can simplify the inversion formula (V.12a) to the form

$$F(t,u) = \frac{4}{\pi^2}\,tu\ \overline{K(t,x)\,K(u,z)}_D\,\frac{\partial^2 G(x,z)}{\partial z\,\partial x}$$

$$-\frac{4}{\pi^2}\,tu\int_t^{\phi(u)} \frac{dx}{\sqrt{x^2-t^2}}\,\frac{1}{\sqrt{\psi^2(x)-u^2}}\left[\frac{\partial G(x,z)}{\partial x}\right]_{z=\psi(x)} \qquad \text{(V.15a)}$$

From Eq. (V.10b) we get instead of Eq. (V.12a) the equation

$$F(t,u) = \frac{4}{\pi^2}\, tu \iint\limits_{D(t,u)} \frac{dx\, dz}{\sqrt{(x^2-t^2)(z^2-u^2)}} \frac{\partial^2 G(x,z)}{\partial x\, \partial z}$$

$$+ \frac{4}{\pi^2}\, tu \int_t^{\phi(u)} \frac{dx}{\sqrt{x^2-t^2}} \frac{1}{\sqrt{\psi^2(x)-u^2}} \left[\frac{\partial G(x,z)}{\partial z}\right]_{z=\psi(x)} \frac{d\psi}{dx}. \qquad \text{(V.12b)}$$

Since G(x,z) is a regular function, the order of derivation can be interchanged, i.e.

$$\frac{\partial^2 G(x,z)}{\partial t\, \partial x} = \frac{\partial^2 G(z,x)}{\partial x\, \partial z}$$

and hence the first terms on the right-hand side of Eqs. (V.12a,b) and (V.15a) are identical. The comparison of Eqs. (V.12b) with (V.15a) yields the interesting equality

$$-\int_t^{\phi(u)} \frac{dx}{\sqrt{x^2-t^2}} \frac{1}{\sqrt{\psi^2(x)-u^2}} \left[\frac{\partial G(x,z)}{\partial x}\right]_{z=\psi(x)} = \int_t^{\phi(u)} \frac{dx}{\sqrt{x^2-t^2}} \frac{1}{\sqrt{\psi^2(x)-u^2}} \left[\frac{\partial G(x,z)}{\partial z}\right]_{z=\psi(x)} \frac{d\psi}{dx}$$

for all $t \in \{0,\phi(u)\}$. We can therefore make the conclusion:

$$\frac{dG}{dx} = \left[\frac{\partial G(x,z)}{\partial x}\right]_{z=\psi(x)} + \left[\frac{\partial G(x,z)}{\partial z}\right]_{z=\psi(x)} \frac{d\psi}{dx} = 0\ . \qquad \text{(V.16)}$$

From this it follows that G(x,z) has to be constant along the boundary $z = \Psi(x)$ of the domain D. This fact must be remembered if we search for the transformed F(t,u) of an arbitrary function G(x,z), because if G(x,z) does not fulfil the boundary condition

$$G[x,\psi(x)] = G[\phi(z),z] = \text{const} \qquad \text{(V.17)}$$

an inverse will not exist in a unique sense. If G(x,z) satisfies this condition together with the suitable conditions for its differentiability in the compact domain D, we have a unique inverse which can be calculated from one of the equations (V.10a,b), (V.12a,b), or (V.15a,b). The last equation is given by

$$F(t,u) = \frac{4}{\pi^2}\, tu\ \overline{K(t,x)\, K(u,z)}_D\ \frac{\partial^2 G(x,z)}{\partial x\, \partial z}$$

$$- \frac{4}{\pi^2}\, tu \int_u^{\psi(t)} \frac{dz}{\sqrt{z^2-u^2}} \frac{1}{\sqrt{\phi^2(z)-t^2}} \left[\frac{\partial G(x,z)}{\partial z}\right]_{x=\phi(z)} \ . \qquad \text{(V.15b)}$$

In order to get a short notation, we will denote the inverse operator by

$$\overline{K(t,x)\,K(u,z)}_D^{-1}\;\overline{K(x,t)\,K(z,u)}_D = I\ .$$

With this notation we can rewrite the relations between $\rho(x,z)$ and $g(a,b)$ from Table II.2 in the general form

$$\rho(x,z) = \frac{1}{\pi^2}\,\overline{K(x,t)\,K(z,u)}_D\,g(a,b) \tag{V.18}$$

$$g(a,b) = \pi^2\,\overline{K(a,x)\,K(b,z)}_D^{-1}\,\rho(x,z)\ . \tag{V.19}$$

The other two-dimensional relations must be changed in an analogous way.

Example

Let us assume the normalized transverse density distribution

$$\rho(x,z) = \begin{cases} \frac{2}{\pi AB}\left[1-\left(\frac{x}{A}\right)^2-\left(\frac{z}{B}\right)^2\right] & \text{inside the ellipse } \frac{x^2}{A^2}+\frac{z^2}{B^2}=1 \\ 0 & \text{outside this ellipse .} \end{cases} \tag{V.20}$$

The boundary function is given by

$$B(x,z) = 1-\left(\frac{x}{A}\right)^2-\left(\frac{z}{B}\right)^2 \tag{V.21}$$

and we see immediately that $\rho(x,z)$ is constant and zero at this boundary. Since $\rho(x,z)$ is regular and bounded inside the compact set D:

$$D = \{x,z\,|\,0\le x,\ 0\le z,\ B(x,z)\ge 0\}$$

an inverse $\overline{K(a,x)K(b,z)}_D^{-1}\,\rho(x,z)$ will exist. First, we determine the functions ρ and Ψ defined in Eqs. (V.2):

$$\begin{aligned} B[\phi(z),z] = 0 &\Rightarrow \phi(z) = A\,\sqrt[+]{1-\left(\frac{z}{B}\right)^2} \\ B[x,\psi(x)] = 0 &\Rightarrow \psi(x) = B\,\sqrt[+]{1-\left(\frac{x}{A}\right)^2}\ . \end{aligned} \tag{V.22}$$

Now, we shall calculate the connected amplitude distribution $g(a,b)$ with the help of Eqs. (V.19) and (V.15b):

$$g(a,b) = 4ab\left\{\iint\limits_{D(a,b)} \frac{dx\,dz}{\sqrt{x^2-a^2}\,\sqrt{z^2-b^2}}\,\frac{\partial^2\rho}{\partial x\,\partial z} - \int_b^{\psi(a)} \frac{dz}{\sqrt{z^2-b^2}}\,\frac{1}{\sqrt{\phi^2(z)-a^2}}\,\frac{\partial\rho(x,z)}{\partial z}\bigg|_{x=\phi(z)}\right\}\ . \tag{V.23}$$

For the derivatives of ρ we get

$$\frac{\partial\rho}{\partial z} = -\frac{4}{\pi AB}\frac{z}{B^2} \qquad \frac{\partial^2\rho}{\partial x\,\partial z} = 0$$

and Eq. (V.23) reduces to

$$g(a,b) = +\frac{16ab}{\pi AB^3}\int_b^{\psi(a)} \frac{dz}{\sqrt{z^2-b^2}} \quad \frac{z}{\sqrt{\phi^2(z)-a^2}} \;. \tag{V.24}$$

Using the substitutions

$$t = z^2, \quad \alpha = b^2, \quad \beta = \psi^2(a) = B^2\left[1-\left(\frac{a}{A}\right)^2\right]$$

we obtain for g(a,b) the expressions:

$$g(a,b) = \frac{8ab}{\pi A^2B^2}\int_\alpha^\beta \frac{dt}{\sqrt{t-\alpha}\,\sqrt{\beta-t}} = \frac{8ab}{A^2B^2} \tag{V.25}$$

for

$$a, b \in D = \{a, b \mid 0 \le a,\ 0 \le b,\ B(a,b) \ge 0\} \;.$$

The proof that g(a,b) is normalized over D can be done easily, substituting $a/A = r\cos\theta$, $b/B = r\sin\theta$ into the integral

$$\frac{8}{A^2B^2}\iint_D ab\; da\; db = 8\int_0^{\pi/2} dz\,\sin z\,\cos z\int_0^1 r^3\,dr = 1 \;.$$